Solutions (Phase Test 4 : Code-1831)

Mathematics

Solutions (Phase test 4)

(61) A: 1, 2, 3, 4, 5   B: x, y, z

A to B is onto if

(i) 3 elements of A have same image & remaining two have distinct images

No. of ways $= {}^5C_3 \times 3 \times 2 \times$ → selecting 3 elements which have same image.

(ii) Two elements of A have same image, other two have same image & 5th have third element of B as it's image

No. of ways $= \dfrac{{}^5C_2 \times {}^3C_2}{2!} \times 3 \times 2 \times 1$ → making two identical group of 2 elements each.

Alt → Total ~~of~~ no. of function − no. of into functions.

(62) $2x - x^2 > 0$ —① & $\log(2x - x^2) \geq 0$ —②

$\Rightarrow \log(2x - x^2) \geq \log 1$

$\Rightarrow 2x - x^2 \geq 1$

common of ① & ② is $2x - x^2 \geq 1 \Rightarrow x^2 - 2x + 1 \leq 0$

$\Rightarrow (x-1)^2 \leq 0$

$\Rightarrow x = 1$

(63) Since $f(1) = f(2) = f(3) = 0 \Rightarrow$ many one

$f(x)$ is cubic polynomial

as $x \to \infty \Rightarrow f(x) \to \infty$

& $x \to -\infty \Rightarrow f(x) \to -\infty$

Hence range $= (-\infty, \infty) \Rightarrow$ Onto function

(64) $x^2 - [x]^2 \geq 0 \Rightarrow (x - [x])(x + [x]) \geq 0$

Since $x - [x] = \{x\} \geq 0$ ∴ it is zero for all integer so

$(x-[x])\,(x+[x]) \geq 0$

$\Rightarrow \quad x+[x] \geq 0 \quad$ or $\quad x \in$ integer.

$\Rightarrow \quad x \in R^{+} \quad$ or $\quad x \in$ integer

(because x & $[x]$ are simultaneously +ive or -ive)

so $\quad x \in [0,\infty) \cup \{I\}$

(65) $\quad f(x) = \dfrac{x(x-1)}{x(x+2)} = \dfrac{x-1}{x+2}, \quad x \neq 0,-2$

let $\quad y = \dfrac{x-1}{x+2} \Rightarrow yx + 2y = x-1$

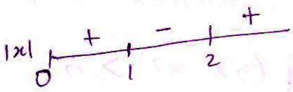
$\Rightarrow (y-1)x = -1-2y$

$x = \dfrac{2y+1}{1-y} = f^{-1}(y)$

so $\quad f^{-1}(x) = \dfrac{2x+1}{1-x}$

$\dfrac{d}{dx} f^{-1}(x) = \dfrac{3}{(1-x)^2}$

(66) $\quad \dfrac{1-|x|}{2-|x|} \geq 0 \Rightarrow \dfrac{|x|-1}{|x|-2} \geq 0$



$\Rightarrow |x| \in [0,1] \cup (2,\infty)$

$\Rightarrow x \in [0,1] \cup (2,\infty) \cup [-1,0] \cup (-\infty,-2)$

$\Rightarrow x \in (-\infty,-2) \cup [-1,1] \cup (2,\infty)$

(67) $\quad x$ should be integer.

$16-x \geq 2x-1 \quad$ & $\quad 20-3x \geq 4x-5$

$\Rightarrow 17 \geq 3x \qquad 25 \geq 7x$

$\Rightarrow x \leq 17/3 \qquad x \leq \dfrac{25}{7}$

$\Rightarrow \quad 16-x,\ 2x-1,\ 20-3x,\ 4x-5$ should be +ive integer

$\Rightarrow \quad \dfrac{5}{4} \leq x \leq 20/3$

so possible x can be $\quad \dfrac{5}{4} \leq x \leq \dfrac{25}{7}$

$\Rightarrow x = 2, 3$

(68) $\quad n(A \times B) = 12$ elements

no of subsets of $A \times B = 2^{12}$

Every sub-set of $A \times B$ is relation so total relations $= 2^{12}$

69 (a) $y = \pm\sqrt{4ax}$ for $-$ive values of $4ax$ there is no real y so not a function

(b) $y = |x|$ is function (each & every real value of x has one & only one image)

(c) $y = \pm\sqrt{1-x^2}$ Not a function

(d) $y = \pm\sqrt{x^2-1}$ Not a function.

70 Let $y = \frac{x^2+1}{x^2+2} \Rightarrow yx^2 + 2y = x^2 + 1$

$\Rightarrow x^2 = \frac{1-2y}{y-1} \geq 0$

$\Rightarrow \frac{1}{2} \leq y < 1$

$\Rightarrow \frac{1}{2} \leq \frac{x^2+1}{x^2+2} < 1$

$\Rightarrow \pi/6 \leq \sin^{-1}\left(\frac{x^2+1}{x^2+1}\right) < \pi/2$

71 $y = \frac{x^2+x+2}{x^2+x+1}$

$(y-1)x^2 + (y-1)x + y-2 = 0$

$\Rightarrow D \geq 0 \Rightarrow (y-1)^2 - 4(y-1)(y-2) \geq 0$

$\Rightarrow (y-1)[y-1-4y+8] \geq 0$

$\Rightarrow (y-1)[7-3y] \geq 0$

$\Rightarrow 1 \leq y \leq 7/3$

But ~~from~~ $\frac{x^2+x+2}{x^2+x+1} \neq 0 \Rightarrow y \neq 1$

72 $f(x) = y = \log_a\left(x+\sqrt{x^2+1}\right)$

Since $\sqrt{x^2+1} > |x| \Rightarrow f(x)$ is defined for all real values of x.

$y = \log_a\left(x+\sqrt{x^2+1}\right) \Rightarrow a^y = x+\sqrt{x^2+1}$

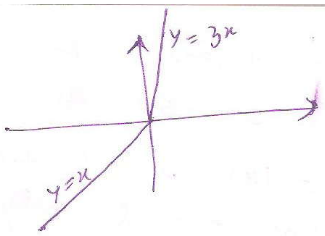
$(a^y - x)^2 = x^2+1$

$\Rightarrow a^{2y} - 2a^y x + \cancel{x^2} = \cancel{x^2}+1$

$\Rightarrow 2a^y x = a^{2y} - 1$

$\Rightarrow x = \frac{a^y - a^{-y}}{2} = f^{-1}(y)$

(73) $f(x) = \begin{cases} x, & x \le 0 \\ 3x, & x \ge 0 \end{cases}$



Graph of y = f(x)

From graph onto & one to one functions

(74) $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \frac{1}{2}$ & $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & r^2 & r^2 \end{vmatrix} = 4$

$$\begin{vmatrix} 1+a^2b^2+2ab & 1+b^2p^2+2bp & 1+c^2p^2+2cp \\ 1+a^2q^2+2aq & 1+b^2q^2+2bq & 1+c^2q^2+2cq \\ 1+a^2r^2+2ar & 1+b^2r^2+2br & 1+c^2r^2+2cr \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b^2 & 2b \\ 1 & q^2 & 2q \\ 1 & r^2 & 2r \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ 2a & 2b & 2c \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ p^2 & q^2 & r^2 \\ b & q & r \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a & b & c \end{vmatrix}$$

$$= 2\left(-\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{vmatrix}\right)\left(-\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}\right)$$

$$= 2 \times 8 \times 1 = 16$$

(75) $\Delta = -2(3\cos\theta + 4\sin\theta) - (\tan\theta + \sec^2\theta)(-3\sin\theta + 3\sin\theta)$
$+ 3(4\sin\theta + 3\cos\theta)$

$= 4\sin\theta + 3\cos\theta$, $\theta \in [0, \pi/2]$

$= 5\left[\sin\theta\,(4/5) + 3/5\cos\theta\right] = 5\sin(\theta + \phi)$

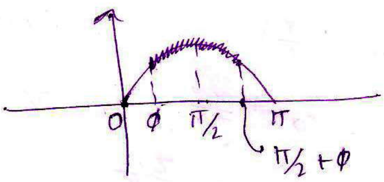
where $\phi = \cos^{-1} 4/5 = \sin^{-1} 3/5$

if $\theta \in [0, \pi/2]$
$\theta + \phi \in [\phi, \pi/2 + \phi]$

Graph of $y = \sin x$

if $x \in \left[\phi, \frac{\pi}{2}+\phi\right]$

then it is shaded portion of curve



~~Max~~ Maximum value of $\sin x$ in the interval is 1 & Minimum is either at ϕ or $\frac{\pi}{2}+\phi$

$\sin\phi = \sin\sin^{-1}\frac{3}{5} = \frac{3}{5}$

$\sin\left(\frac{\pi}{2}+\phi\right) = \cos\phi = \cos\sin^{-1}\frac{3}{5} = \frac{4}{5}$

so minimum $= \frac{3}{5}$

Hence $5\sin(\theta+\phi) \in [3, 5]$

(76) $\cot\left[\sum_{n=1}^{19}\cot^{-1}\left(1+\frac{2n(n+1)}{2}\right)\right]$

$= \cot\left[\sum_{n=1}^{19}\cot^{-1}(1+n^2+n)\right] = \cot\left[\sum_{n=1}^{19}\tan^{-1}\frac{1}{1+n^2+n}\right]$

$= \cot\left[\sum_{n=1}^{19}\tan^{-1}\frac{n+1-n}{1+(n+1)n}\right] = \cot\left[\sum_{n=1}^{19}\left[\tan^{-1}(n+1)-\tan^{-1}n\right]\right]$

$= \cot\left[\tan^{-1}20-\tan^{-1}1\right] = \left[\cot\tan^{-1}\frac{20-1}{1+20\cdot1}\right] = \cot\tan^{-1}\frac{19}{21}$

$= \frac{21}{19}$

(77) let $\cos^{-1}\frac{2}{3} = \theta \Rightarrow \cos\theta = \frac{2}{3}$

$\Rightarrow \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} = \frac{2}{3} \Rightarrow \frac{2}{2\tan^2\theta/2} = \frac{5}{1}$

$\Rightarrow \tan\theta/2 = \pm\frac{1}{\sqrt{5}}$

Since $\theta \in (0, \pi/2) \Rightarrow \tan\theta/2 = \frac{1}{\sqrt{5}}$

(78) $\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$

Since $\frac{yz}{xr}\cdot\frac{xz}{yr} = \frac{z^2}{r^2} < 1$

$= \tan^{-1}\left(\frac{\frac{yz}{xr}+\frac{xz}{yr}}{1-\frac{yz}{xr}\cdot\frac{xz}{yr}}\right)$

$= \tan^{-1}\left(\frac{\frac{z(y^2+x^2)}{xyr^2}}{1-\frac{z^2}{r^2}}\right) = \tan^{-1}\left(\frac{z(y^2+x^2)}{(r^2-z^2)}\cdot\frac{r^2}{xyr^2}\right)$

$\mathrm{Tan}^{-1}\left(\frac{zr}{xy}\right)$ because $r^2 - z^2 = x^2 + y^2$

$= \pi/2 - \cot^{-1}\frac{xy}{zr} = \pi/2 - \mathrm{Tan}^{-1}\frac{zr}{xy}$

so $\phi = \frac{zr}{xy}$

(79) $f(x) = y = \mathrm{Tan}^{-1}\sqrt{\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 - \sin x/2)^2}}$

$= \mathrm{Tan}^{-1}\frac{|\cos x/2 + \sin x/2|}{|\cos x/2 - \sin x/2|}$

Since $x \in (0, \pi/2)$
$\Rightarrow x/2 \in (0, \pi/4)$
so $\cos x/2 > \sin x/2$

$= \mathrm{Tan}^{-1}\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}$

$y = \mathrm{Tan}^{-1}\tan(\pi/4 + x/2) = \pi/4 + x/2$

So $y = f(x)$ is straight line ~~same~~ slope -2
line normal to above have
& when $x = \pi/6$, $y = \pi/3$ so eqn. of normal is

$y - \pi/3 = -2(x - \pi/6)$

$2x + y = 2\pi/3 \Rightarrow$ alternative (d) satisfies it.

(80) Expand both determinants.

(81) Sub $x = 0$ both sides & get g

(82) Homogenious linear eqn. have unique sol. $\Rightarrow$ only trivial solutions.

$\Rightarrow \Delta \neq 0 \Rightarrow \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} \neq 0$

(83) $P^3 = Q^3$ —① & $P^2Q = Q^2P$ —②

① − ②

$\Rightarrow P^3 - P^2Q = Q^3 - Q^2P$

$\Rightarrow P^2(P-Q) = Q^2(Q-P) \Rightarrow P^2(P-Q) + Q^2(P-Q) = 0$

$\Rightarrow (P^2 + Q^2)(P-Q) = 0$

$(P-Q)(P^2+Q^2)=0$

$\Rightarrow |(P-Q)(P^2+Q^2)|=0 \Rightarrow |P-Q||P^2+Q^2|=0$

$\Rightarrow |P^2+Q^2|=0$

(84) Statement (i) $\text{adj}(\text{adj}A) = |A|^{n-2}A$

$= |A|^0 A = A$

is correct.

Statement (ii) $|\text{adj}A| = |A|^{2-1} = |A|$ is false.

(85) $R_1 \to R_1+R_2+R_3$

$$(2\sin x+\cos x)\begin{vmatrix} 1 & \sin x & \sin x \\ 1 & \cos x & \sin x \\ 1 & \sin x & \cos x \end{vmatrix}=0$$

$R_2 \to R_2-R_1$ & $R_3 \to R_3-R_1$

$$(2\sin x+\cos x)\begin{vmatrix} 1 & \sin x & \sin x \\ 0 & \cos x-\sin x & 0 \\ 0 & 0 & \cos x-\sin x \end{vmatrix}=0$$

$\Rightarrow (2\sin x+\cos x)(\cos x-\sin x)^2=0$

$\Rightarrow 2\sin x+\cos x=0$ or $\cos x-\sin x=0$

$\Rightarrow \tan x=-\frac{1}{2}$ or $\tan x=1$

if $x\in\left[\frac{\pi}{4},\frac{\pi}{4}\right]$ $\tan x=-\frac{1}{2}$ once in $\left[-\frac{\pi}{4},0\right]$

& $\tan x=1$ at $x=\frac{\pi}{4}$

$\Rightarrow$ Two sol.

(86) Since $\sin(A+B)\sin(A-B) = \sin^2 A-\sin^2 B$

so given quantity is $\sin^2\sin^{-1}(0.5) - \sin^2\sin^{-1}(0.4)$

$= (0.5)^2-(0.4)^2 = 0.09$

$= \frac{9}{100}$

(87) $$\lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0}\frac{1}{x^2}\begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$$

$$= \lim_{x\to 0}\frac{x}{x^2}\begin{vmatrix} \sin x & \cos x & \tan x \\ x^2 & x & 1 \\ 2x & 1 & 1 \end{vmatrix}$$

(Taking x common from second row)

$$= \lim_{x\to 0}\begin{vmatrix} \frac{\sin x}{x} & \cos x & \tan x \\ x & x^2 & x \\ 2 & 1 & 1 \end{vmatrix}$$ (dividing first row by x)

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

(88) $\Delta = \begin{vmatrix} a & 2 & 5 \\ 2 & 1 & 3 \\ 0 & 3 & 7 \end{vmatrix} = -2a - 2(14-15) = -2a+2$

if $a \neq 1 \Rightarrow$ unique sol.

if $a = 1$ then $\Delta = 0$ & Eqn. are

$x + 2y + 5z = 1$ — (1)

$2x + y + 3z = 1$ — (2) $\Rightarrow 2\times(1) - (2) \Rightarrow (3)$

$3y + 7z = 1$ — (3) $\Rightarrow$ Eqn. are dependent Eqn.

so when $a = 1$ Eqn. have infinite sol $\Rightarrow$ for any value of a Eqn. is consistent

(89) $A + B = 2B'$ — (1) & $3A + 2B = I_3$ — (2)

Taking transpose $\Rightarrow A' + B' = 2B$ — (3) & $3A' + 2B' = I$ — (4)

eliminate A' & B' & form with the help of (1), (3), (4) Eqn. in A & B

From (3) $A' = 2B - B'$ sub in (4)

$\Rightarrow 3(2B - B') + 2B' = I$

$\Rightarrow 6B - 3B' + 2B' = I$ sub. B' from (1)

$\Rightarrow 6B - B' = I$

$\Rightarrow 6B - \left(\frac{A+B}{2}\right) = I$

$\Rightarrow \frac{11B}{2} - A/2 = I$ — (5)

Now solve (2) & (5) ~~Are~~ for A & B

$\Rightarrow A = I/5$ & $B = I/5$

Now check the alternatives.