Solutions (Class Test 10 : Code-1833)

Mathematics

Class test - 10

Q61 $\lim_{x\to 0} \frac{kx - 3\tan^{-1}x}{3\sin^{-1}x} = \lim_{x\to 0} \frac{k - 3\frac{\tan^{-1}x}{x}}{3\frac{\sin^{-1}x}{x}}$

$= \frac{k-3}{3}$

$\lim_{x\to 0} \frac{x^2-16}{x-4} = 4$

Hence $\lim_{x\to 0} \left(\frac{kx - 3\tan^{-1}x}{3\sin^{-1}x}\right)^{\frac{x^2-16}{x-4}} = \left(\frac{k-3}{3}\right)^4$

$\Rightarrow \left(\frac{k-3}{3}\right)^4 = \frac{1}{81}$

$\Rightarrow \frac{k-3}{3} = \pm\frac{1}{3} \Rightarrow k = 4, 2$

if $k=2$ then $\lim_{x\to 0} \frac{kx-3\tan^{-1}x}{3\sin^{-1}x} = -\frac{1}{3}$

$\Rightarrow$ as $x\to 0$ $\frac{kx-3\tan^{-1}x}{3\sin^{-1}x}$ is $-$ive

so $\left(\frac{kx-3\tan^{-1}x}{3\sin^{-1}x}\right)^{\frac{x^2-16}{x-4}}$ is not defined

Hence $k=4$ only.

62 R.H.L.

$\lim_{x\to 0^+} \frac{\tan(\{x\}-1)\sin\{x\}}{\{x\}(\{x\}-1)}$

$\lim_{x\to 0^+} \frac{\tan(x-[x]-1)\sin(x-[x])}{(x-[x])(x-[x]-1)}$

$\lim_{x\to 0^+} \frac{\tan(x-1)\sin x}{x(x-1)} = \lim_{x\to 0}\frac{\tan(x-1)}{x-1}$

$= \tan 1$

L.H.L.
$$\lim_{x\to 0^-} \frac{\tan(x-[x]-1)\ \sin(x-[x])}{(x-[x])\ (x-[x]-1)}$$

$$\lim_{x\to 0^-} \frac{\tan(x+1-1)\ \sin(x+1)}{(x+1)\ (x+1-1)}$$

$$\lim_{x\to 0^-} \frac{\sin(x+1)}{x+1} = \sin 1$$

$\Rightarrow$ given limit does not exist.

Q63 $\lim_{h\to 0} \dfrac{f(2+2h+h^2) - f(2)}{f(1+h+h^2) - f(1)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$\lim_{h\to 0} \frac{[f'(2+2h+h^2)](2+2h) - 0}{[f'(1+h+h^2)](1+2h)}$$

$$= \frac{2f'(2)}{f'(1)}$$

(64) $\lim_{x\to 0} \frac{1}{x}\tan^{-1}2x = \lim_{x\to 0} 2\left(\frac{\tan^{-1}2x}{2x}\right) = 2$

(65) $\lim_{x\to \pi/4} \dfrac{\sqrt{1-\sin x}}{\pi - 2x} = \lim_{x\to \pi/4} \dfrac{|\sin x/2 - \cos x/2|}{\pi - 2x}$

$$\lim_{x\to \pi/4} \frac{\sqrt{2}\,|\sin(x/2 - \pi/4)|}{\pi - 2x}$$

L.H.L. $\lim_{x\to \pi/4^-} \dfrac{-\sqrt{2}[\sin(x/2 - \pi/4)]}{-4(x/2 - \pi/4)} = \dfrac{-\sqrt{2}}{-4}$

$$= \frac{1}{2\sqrt{2}}$$

Similarly R.H.L. $= -\dfrac{1}{2\sqrt{2}}$ $\Rightarrow$ given limit does not exist.

(66) $\lim_{x\to 0}\left(1-\tan^2\sqrt{x}\right)^{1/x}$

$= e^{\lim_{x\to 0} -(\tan^2\sqrt{x})\times\frac{1}{x}} = e^{\lim_{x\to 0} -\frac{\tan^2\sqrt{x}}{(\sqrt{x})^2}}$

$= e^{-1}$

(67) $\lim_{\alpha\to 0} \frac{e\left(e^{\cos\alpha^n-1}-1\right)}{\alpha^m(\cos\alpha^n-1)}(\cos\alpha^n-1)$

$= \lim_{\alpha\to 0} \frac{e(\cos\alpha^n-1)}{\alpha^m}$

$= \lim_{\alpha\to 0} \frac{-2e\sin^2\left(\frac{\alpha^n}{2}\right)}{\alpha^m\cdot\left(\frac{\alpha^n}{2}\right)^2}\left(\frac{\alpha^n}{2}\right)^2$

$= \lim_{\alpha\to 0} \frac{-2}{4}e\cdot\frac{\alpha^{2n}}{\alpha^m}$

$= -\frac{1}{2}e\,\alpha^{2n-m}$

For limit to be $-e/2 \Rightarrow 2n-m=0$

hence $\frac{m}{n}=2$

(68) $\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}=2$

$\frac{\sqrt{b}-2}{0} \Rightarrow \sqrt{b}=2 \Rightarrow b=4$

if b is not 4 then value of limit is not finite

$\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x} = \lim_{x\to 0}\frac{1\times a}{2\sqrt{ax+4}}=2$

$= \frac{a}{2\times 2}=2$

$\Rightarrow a = 8$

(69) $\lim_{x\to 0}\left(\cos x + a^3\sin b^6x\right)^{1/x} = e^{512}$

$= e^{\lim_{x\to 0}\left(\cos x + a^3\sin b^6x - 1\right)\times\frac{1}{x}}$

$= e^{\lim_{x\to 0}\frac{\cos x - 1}{x} + a^3\frac{\sin b^6x}{x\cdot b^6}\cdot(b^6)}$

$= e^{\lim_{x\to 0}\frac{-2\sin^2 x/2}{x} + a^3b^6}$

$= e^{0+a^3b^6} = e^{512}$

$\Rightarrow a^3b^6 = 512 \Rightarrow ab^2 = 64$

(70) $\lim_{x\to 0}\frac{2ax+(a-1)\sin x}{\frac{\tan^3x}{x^3}\cdot x^3}$

$\lim_{x\to 0}\frac{2ax+(a-1)\sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\frac{2a+(a-1)\cos x}{3x^2} = \frac{2a+a-1}{0}$

$\Rightarrow a = \frac{1}{3}$

$\lim_{x\to 0}\frac{2/3 - 2/3\cos x}{3x^2} = \lim_{x\to 0}\frac{2}{9}\left(\frac{1-\cos x}{x^2}\right)$

$= \frac{2}{9}\times\frac{1}{2} = \frac{1}{9}$

$a = \frac{1}{3},\ L = \frac{1}{9} \Rightarrow a+L = \frac{4}{9}$

(71) let $f(x) = a(x+\pi)(x-\pi)$ (roots of $f(x)$ are known)

$f(\pi/2) = a(3\pi/2)(-\pi/2) = -3\pi^2/4$

$\Rightarrow a = 1$

so $f(x) = (x+\pi)(x-\pi)$

$$\lim_{x\to -\pi} \frac{(x+\pi)(x-\pi)}{\sin(\sin x)}$$

$$\lim_{x\to -\pi} \frac{2x}{\cos(\sin x)\cos x} = \frac{-2\pi}{\cos 0 \cos\pi} = 2\pi$$

(72) $$\lim_{x\to 0} \frac{x^a \left(\sin^b x / x^b\right) \cdot x^b}{\left(\frac{\sin x^c}{x^c}\right) x^c}$$

$$\therefore \lim_{x\to 0} \frac{x^{a+b}}{x^c} = \lim_{x\to 0} x^{a+b-c}$$

for limit to be finite non zero $\Rightarrow a+b-c = 0$

(73) $$\lim_{x\to 0} \frac{x^n \sin^n x}{x^n - \sin^n x}$$ is finite non zero

$$\lim_{x\to 0} \frac{x^n \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \infty\right)^n}{x^n - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \infty\right)^n}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \infty\right)^n}{1 - \left(1 - \frac{x^3}{3!} + \frac{x^5}{5!} + \infty\right)^n}$$

(~~Taking~~ cancelling x^n from both numerator & denominator)

in denominator 1 gets cancelled on expanding second term & min. power of x is 3

Hence for limit to be finite nonzero minimum power of x in numerator should be 3 so $n = 3$

(74) Correct Q. is $f(1) = 1$

$\& \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = 5$ then find $f'(1)$

Apply L.H rule.

(75) $\lim_{x\to 0} \frac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x\to 0} e^x\left(\frac{e^{\tan x - x} - 1}{\tan x - x}\right)$

$= e^0 \times 1$

(76) $\lim_{x\to 1} \frac{\ln x}{\cos\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1} \frac{\frac{1}{x}}{-\sin\left(\frac{\pi x}{2}\right)\frac{\pi}{2}}$

$= -\frac{2}{\pi}$

(77) $\lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

$= \frac{f'(a)}{\ }$

$\cot^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \pi/2 - \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$= \pi/2 - 3\tan^{-1}x \quad , \quad x < \frac{1}{\sqrt{3}}$

$\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) = 2\tan^{-1}x \quad , \quad x > 0$

so $f'(x) = -\frac{3}{1+x^2}$

$g'(x) = \frac{2}{1+x^2}$

$\lim_{x\to a} \frac{f'(a)}{g'(a)} = \frac{-\frac{3}{1+a^2}}{\frac{2}{1+a^2}} = -3/2$

~~1/2~~

~~2~~

Q 78 & 79 Apply L.H. rule.

(80) $\lim_{x\to\infty} \dfrac{x^{40}\left(\frac{2}{x}+1\right)^{40} \cdot x^{5}\left(\frac{4}{x}+1\right)^{5}}{x^{45}\left(\frac{2}{x}-1\right)^{45}}$

$= \lim_{x\to\infty} \dfrac{\left(\frac{2}{x}+1\right)^{40}\left(\frac{4}{x}+1\right)^{5}}{\left(\frac{2}{x}-1\right)^{45}} = \dfrac{1\times 1}{-1} = -1$

(81) $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & x \in R-[0,1) \\ 0, & x \in [0,1) \end{cases}$

R.H.L. $\lim_{x\to 0^+} f(x) = 0$

L.H.L. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\sin[x]}{[x]}$

$= \dfrac{\sin(-1)}{-1} = \sin 1$

$\Rightarrow$ Given limit does not exist

(82) $\lim_{x\to\infty} x\left[\tan^{-1}\dfrac{x+1}{x+2} - \tan^{-1}\dfrac{x}{x+2}\right]$

$\lim_{x\to\infty} x\left[\tan^{-1}\dfrac{\frac{x+1}{x+2}-\frac{x}{x+2}}{1+\frac{(x+1)x}{(x+2)^2}}\right]$

$= \lim_{x\to\infty} x\left[\tan^{-1}\dfrac{x+2}{(x+2)^2+x(x+1)}\right]$

$= \lim_{x\to\infty} x\left[\tan^{-1}\dfrac{x+2}{2x^2+5x+4}\right]$

as $x \to \infty$

$(\infty \times 0)$

$$= \text{Lt}_{x\to\infty}\; x \tan^{-1}\left(\frac{1+2/x}{2x+5+4/x}\right)$$

$$= \text{Lt}_{x\to\infty}\; x \frac{\tan^{-1}\left(\frac{1+2/x}{2x+5+4/x}\right)}{\frac{1+2/x}{2x+5+4/x^2}} = 1 \quad \left(\frac{1+2/x}{2x+5+4/x}\right)$$

$$= \text{Lt}_{x\to\infty}\; \frac{x(1+2/x)}{2x+5+4/x} = \text{Lt}_{x\to\infty} \frac{1+2/x}{2+5/x+4/x^2} = \frac{1}{2}$$

(Q83) $\text{Lt}_{x\to 1} \dfrac{x\sin(x-[x])}{x-1}$

R.H.L.

$$\text{Lt}_{x\to 1^+} \frac{x\sin(x-1)}{x-1} = 1$$

L.H.L.

$$\text{Lt}_{x\to 1^-} \frac{x\sin(x-0)}{x-1} = \frac{1\times 1}{0^-} = -\infty$$

$\Rightarrow$ limit does not exist.

(84) If $x \to \infty$ then $[x] \equiv x$

$$\text{Lt}_{x\to\infty} \frac{\log x^n - [x]}{[x]}$$

$$= \text{Lt}_{x\to\infty} \frac{n \ln x - x}{x}$$

$$= \text{Lt}_{x\to\infty} \frac{n \ln x}{x} - 1$$

$$= \text{Lt}_{x\to\infty} \frac{n \cdot \frac{1}{x}}{1} - 1 = 0 - 1 = -1$$

R.H.L.

(85) $\lim_{x\to 0+} \dfrac{\sin[\cos x]}{1+[\cos x]} = \lim_{x\to 0+} \dfrac{\sin 0}{1+0} = 0$

L.H.L. $\lim_{x\to 0-} \dfrac{\sin[\cos x]}{1+[\cos x]} = \dfrac{\sin 0}{1+0} = 0$

(86) $\lim_{x\to 0} \dfrac{\sin 3x + ax + bx^3}{x^3} = 0$

$\lim_{x\to 0} \dfrac{3\cos 3x + a + 3bx^2}{3x^2} = \left(\dfrac{3+a}{0}\right)$

$\Rightarrow a = -3$

$\lim_{x\to 0} \dfrac{3\cos 3x - 3 + 3bx^2}{3x^2}$

$\lim_{x\to 0} \dfrac{-9\sin 3x + 6bx}{6x}$

$\lim_{x\to 0} \dfrac{-27\cos 3x + 6b}{6} = -\dfrac{27}{6} + b = 0$

$b = \dfrac{27}{6} = \dfrac{9}{2}$

(87) $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\alpha} + \sin\frac{1}{n} = \left(\frac{1}{1+1/n}\right)^{\alpha} + 0 = 1$

so $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^{\alpha} + \sin\frac{1}{n}\right)^{n} = e^{\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{\alpha} + \sin\frac{1}{n} - 1\right]n}$

let $n = 1/y \Rightarrow e^{\lim_{y\to 0^+} \frac{\left(\frac{1/y}{1/y+1}\right)^{\alpha} + \sin y - 1}{y}}$

$= e^{\lim_{y\to\infty} \frac{\frac{1}{(1+y)^{\alpha}} - 1}{y} + \frac{\sin y}{y}}$

$= e^{\lim_{y\to 0} \frac{-\alpha(1+y)^{-\alpha-1}}{1} + 1} = e^{-\alpha+1}$