## Mathematics Solutions (Application of Derivative)

1. $f'(x) = x(e^x-1)(x-1)(x-2)^3(x-3)^5$

$f'(x) = 0 \Rightarrow x = 0, 1, 2, 3$

look sign of $f'(x)$    $- - - \;|_0\; - - - \;|_1\; + + + \;|_2\; - - \;|_3\; + + +$

$x = 0$ is point of inflection

$x = 1$ " " minima.

$x = 2$ " " maxima

$x = 3$ " " Minima.

2. $f(x) = e^{ax} + e^{-ax} \Rightarrow f'(x) = ae^{ax} - ae^{-ax}$

$= a(e^{ax} - e^{-ax})$

for $f'(x) < 0 \Rightarrow a(e^{ax} - e^{-ax}) < 0$

$\Rightarrow e^{ax} - e^{-ax} > 0$ (because $a<0$)

$\Rightarrow e^{ax} > e^{-ax}$

$\Rightarrow ax > -ax$

$\Rightarrow 2ax > 0 \Rightarrow x < 0$

3. for point of intersection $x^2y = 1-y$ & $xy = 1-y$

$\Rightarrow x^2y = xy \Rightarrow x = 0, y = 0, x = 1$

if $x = 0$, $xy = 1-y \Rightarrow 0 = 1-y \Rightarrow y = 1$

if $x = 1 \Rightarrow xy = 1-y \Rightarrow y = 1-y \Rightarrow y = \frac{1}{2}$

if $y = 0$, $xy = 1-y \Rightarrow 0 = 1 \Rightarrow$ not possible.

so point of intersection are $(0,1)$ & $(1,\frac{1}{2})$

$x^2y = 1-y$, differentiating w.r.to $x$

$\Rightarrow 2xy + x^2\frac{dy}{dx} = -\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-2xy}{x^2+1}$

$\left.\frac{dy}{dx}\right|_{(0,1)} = 0$ & $\left.\frac{dy}{dx}\right|_{(1,\frac{1}{2})} = \frac{-2\times1\times\frac{1}{2}}{1+1} = -\frac{1}{2}$

(1)

Eqn of tangents at these points:

at $(0,1)$ $\Rightarrow$ $y-1 = 0(x-0)$ $\Rightarrow y = 1$ — ①

& at $(1, \frac{1}{2})$ $\Rightarrow y - \frac{1}{2} = -\frac{1}{2}(x-1)$ — ②

Intersection ① & ② $x = 0$ & $y = 1$

Q4 $f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x)$

$= 4\sin x \cos x(\sin^2 x - \cos^2 x)$

$= 2\sin 2x(-\cos 2x) = -\sin 4x$

for $f(x) \uparrow \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0$

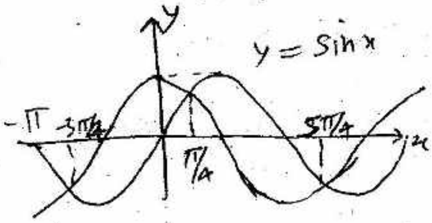
$\Rightarrow 4x \in (\pi, 2\pi)$

$\Rightarrow x \in (\pi/4, \pi/2)$

so (b) is correct because (b) is subset of given solution.

⑤ $f'(x) = \dfrac{\cos x - \sin x}{1+(\sin x + \cos x)^2} > 0$

$\Rightarrow \cos x - \sin x > 0$




From the graph $\cos x > \sin x$ when

$x \in (-3\pi/4, \pi/4) \cup (\frac{5\pi}{4}, 2\pi)$

So correct alternative is (d)

Q6 $f'(x) = 25x^{24}(1-x)^{75} + x^{25} \times 75(1-x)^{74}(-1)$

$= 25x^{24}(1-x)^{74}[1-x-3x] = 25x^{24}(1-x)^{74}(1-4$

$f'(x) = 0 \Rightarrow x = 0, 1, \frac{1}{4}$

$x = 0, 1$ sign of $f'(x)$ ~~does~~ does not change $\Rightarrow$ pt. of inflexi

while $x = \frac{1}{4}$ $f'(x)$ changes sign from +ive to −ive $\Rightarrow$ pt. of local max

②

(7) $f'(x) = 6x^2 - 18ax + 12a^2 = 6(x^2 - 3ax + 2a^2)$

$= 6(x-2a)(x-a)$

$\Rightarrow f'(x) = 0 \Rightarrow x = 2a, a$

Sign of $f'(x)$ : $+$ (a) $-$ (2a) $+$

$\Rightarrow x = a$ is point of maxima & $x = 2a$ (point of minima)

$\quad\;\; \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\quad\;\; p \qquad\qquad\qquad\qquad\qquad\qquad\qquad q$

$p^2 = q \Rightarrow a^2 = 2a \Rightarrow a = 2$

(8) $y^3 - 3xy + 2 = 0 \quad = (1)$

$3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{y^2 - x}$

~~For H points~~ For set H, Tangent is $\parallel$ to x axis

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{y}{y^2-x} = 0 \Rightarrow y = 0$

if $y = 0$ from (1) $\Rightarrow 0 - 0 + 2 = 0$

$\Rightarrow$ not possible

$\Rightarrow$ no point on curve where tangent is $\parallel$ to x axis.

For V set $\frac{dy}{dx} = \infty \Rightarrow y^2 - x = 0 \quad - (2)$

from (1) $y^3 - 3xy + 2 = 0$

substituting $x = y^2$

$\Rightarrow y^3 - 3y^3 + 2 = 0 \Rightarrow y = 1 \Rightarrow x = 1$

so V contain only one point $(1,1)$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$-50 = 4\pi \times 5^2 \times \frac{dr}{dt}$$

(10) $f'(x) = 2(1+b^2)x + 2b = 0 \Rightarrow x = -\frac{b}{1+b^2}$

$f''(x) = 2(1+b^2) > 0 \Rightarrow x = -\frac{b}{1+b^2}$ is pt. of minima

$$f_{min} = f\left(-\frac{b}{1+b^2}\right) = \frac{(1+b^2)\,b^2}{(1+b^2)^2} + 2b\left(\frac{-b}{1+b^2}\right) + 1$$

$$= \frac{b^2}{1+b^2} - \frac{2b^2}{1+b^2} + 1 = \frac{1}{1+b^2}$$

For range of $f_{min}$. let $y = \frac{1}{1+b^2}$

$\Rightarrow 1+b^2 = \frac{1}{y}$

$\Rightarrow b^2 = \frac{1}{y} - 1 \geq 0$

$\Rightarrow -\frac{(y-1)}{y} \geq 0$

$\Rightarrow \frac{y-1}{y} \leq 0$

+   −   +

0   1

$\Rightarrow 0 < y \leq 1$

(Q 11) let $a+b = x$ & $c+d = y$ so, $x, y \geq 0$ & $x, y \leq 2$

$x + y = 2$, $M = xy$

$= x(2-x)$

$M' = 2 - 2x = 0 \Rightarrow x = 1$ (pt. of local max)

~~$M(\frac{1}{2}) = \frac{1}{2}$~~ $M(1) = 1$

$M(0) = 0$, $M(2) = 0$

$\Rightarrow$ Maximum value of $M$ is 1 & minimum is zero (D) so $0 \leq M \leq 1$

Q13 ... S ... while R is ... because if ... fn ... decreases then ... derivative is -ive.

Q: $\frac{dy}{dx}$ = ... slope of normal at (3,-) = $-\frac{1}{\left.\frac{dy}{dx}\right|_{3/4}}$

$$= -\frac{1}{f'(3)}$$

$$\Rightarrow -\frac{1}{f'(3)} = \tan(3\pi/4)$$

$$\Rightarrow f'(3) = 1$$

Q14 (2,3) lies on the curve

$\Rightarrow 9 = 8p + q$ — ①

$y^2 = px^3 + q \Rightarrow 2y\frac{dy}{dx} = 3px^2 \Rightarrow \left.\frac{dy}{dx}\right|_{2,3} = \frac{3p \times 4}{2 \times 3} = 2p$

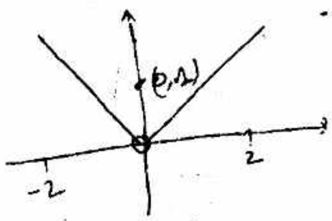
slope of tangent = 4

$\Rightarrow 2p = 4$ — ②

from ① & ② $\Rightarrow p = 2, q = -7$

Q15 ~~Graph of~~ f(x) is discontinuous at $x = 0$

Graph of f(x)




From the graph f(x) has local maxima at $x=1$.

(16) $P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_nx^{2n-1}$

$$= x\left[2a_1 + 4a_2x^2 + \ldots + 2na_nx^{2n-2}\right]$$

$P'(x) = 0 \Rightarrow x = 0$ only because $2a_1 + 4a_2x^2 + 6a_3x^4 + \ldots + 2na_nx^{2n} \neq 0$

Since $a_1, a_2, a_3$ ... are $> 0$ & powers of $x$ are even

$\Rightarrow$ $p'(x) = 0 \Rightarrow x = 0$

& $p''(0) = 2a_1 > 0 \Rightarrow x = 0$ is pt. of local minima.

**Q17** $f'(x) > g'(x)$

Integrating both side from $x_0$ to $x$, $x > x_0$

$$\Rightarrow \int_{x_0}^{x} f'(x)\,dx > \int_{x_0}^{x} g'(x)\,dx$$

$$\Rightarrow f(x) - f(x_0) > g(x) - g(x_0)$$

$$\Rightarrow f(x) > g(x), \quad x > x_0$$

**Q18** $f(x)$ is continuous but non differentiable at $x = 2$

$f'(2^-) = 6x + 12\big|_{x=2} = 18 > 0$

$f'(2^+) = -1$

$\Rightarrow$ $f'(x)$ changes sign from +ive to −ive

$\Rightarrow x = 2$ is point of local maxima

**Q20** $f'(x) = e^{-x} - (x+2)e^{-x} = -(x+1)e^{-x}$

when $-(x+1) > 0 \Rightarrow x < -1 \Rightarrow f(x) \uparrow$

& $-(x+1) < 0 \Rightarrow x > -1 \Rightarrow f(x) \downarrow$

**(21)** $f'(x) = (x-a)^{2n}(x-b)^{2p+1}$

$\Rightarrow f'(x) = 0 \Rightarrow x = a$ & $x = b$

sign of $f'(x)$: $-$ (before $a$), $-$ (between $a$ and $b$), $+$ (after $b$)

$\Rightarrow x = a$ is point of inflection & $x = b$ point of local minima.

(22) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (a-2)^2 - 2(1-a)$

$= a^2 - 2a + 2 = f(a)$

$f'(a) = 2a - 2 = 0 \Rightarrow a = 1$

$f''(a) = 2 > 0 \Rightarrow a=1$ $f(a)$ have ~~total~~ minimum value.

(Q23) $f'(-1) = 0$ & $f'(2) = 0$

$f'(x) = \frac{a}{x} + 2bx + 1$ — ①

$\Rightarrow \frac{a}{-1} - 2b + 1 = 0$ — ②

$\frac{a}{2} + 4b + 1 = 0$

solve for a & b.

(Q24) $f(3) = 0$, $f(4) = 0$, $f(5) = 0$, $f(6) = 0$

By rolles mean value theorem between any two roots of eqn. $f(x) = 0$ there exist at least one root of eqn. $f'(x) = 0$

$\Rightarrow f'(x) = 0$ have at least one root in $(3,4)$ in $(4,5)$, $(5,6)$ — ①

But $f'(x)$ is cubic polynomial $\Rightarrow f'(x) = 0$ have at the most three distinct zero's — ②

Hence from ① & ② $f'(x) = 0$ have exactly one root in each of interval.

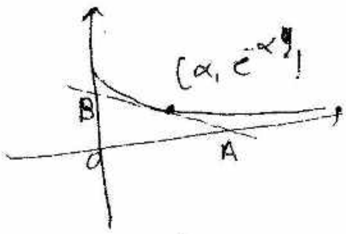
Q25 Curve $y = e^{-|x|}$ is symmetric about y axis. Hence there will be two desired points.

For $x>0$ the curve is $y = e^{-x} \Rightarrow \frac{dy}{dx} = -e^{-x}$

let point be $(\alpha, e^{-\alpha})$

Eqn. of tangent at above point is

$$y - e^{-\alpha} = -e^{-\alpha}(x-\alpha)$$




For y intercept $x = 0$

$$\Rightarrow y_{in} - e^{-\alpha} = -e^{-\alpha}(-\alpha)$$

$$\Rightarrow y_{int} = e^{-\alpha}(1+\alpha)$$

for x intercept $y = 0 \Rightarrow -e^{-\alpha} = -e^{-\alpha}(x_{in} - \alpha)$

$$\Rightarrow x_{int} = 1+\alpha$$

so A is $(1+\alpha, 0)$ & B is $(0, e^{-\alpha}(1+\alpha))$

Area of triangle OAB $= \frac{1}{2}(1+\alpha)\, e^{-\alpha}(1+\alpha)$

$$\Delta = \frac{1}{2} e^{-\alpha}(1+\alpha)^2$$

For maximum area $\frac{d\Delta}{d\alpha} = \frac{1}{2}\left[(1+\alpha)^2(-e^{-\alpha}) + 2e^{-\alpha}(1+\alpha)\right]$

$$= \frac{1}{2}(1+\alpha)e^{-\alpha}[-\alpha - 1 + 2]$$

$$= \frac{1}{2}(\alpha+1)e^{-\alpha}(1-\alpha)$$

$\frac{d\Delta}{d\alpha} = 0 \Rightarrow \alpha = 1$, $\alpha > 0$

& $\alpha = 1$ is point of maxima for $\Delta$

so point on the curve is $(1, e^{-1})$

On left of y axis point is $(-1, e^{-1})$ (curve is symm about y axis)

$\Rightarrow$

(26) Distance from origin $= \sqrt{x^2+y^2}$

$= \sqrt{\left(a\sin t - b\sin\frac{at}{b}\right)^2 + \left(a\cos t - b\cos\frac{at}{b}\right)^2}$

$= \sqrt{a^2+b^2 - 2ab\left[\cos t\cdot\sin\frac{at}{b} + \sin t\sin\frac{at}{b}\right]}$

$d = \sqrt{a^2+b^2 - 2ab\cos\left(t+\frac{at}{b}\right)}$

$-\cos\left(t+\frac{at}{b}\right) \le 1$

so $d_{max} = \sqrt{a^2+b^2+2ab} = a+b$

(28) slope of line $= -\frac{b}{a}$

$y = b e^{-x/a}$

$\frac{dy}{dx} = -\frac{b}{a} e^{-x/a}$

$-\frac{b}{a} e^{-x/a} = -b/a \Rightarrow x = 0$

$y = b e^{-x/a}$ if $x=0 \Rightarrow y = b$

so point of contact $= (0, b)$

Q29 $ax^2 + by^2 = 1$

$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by} = m_1$

similarly $m_2 = -\frac{AX}{BY}$

$m_1 m_2 = -1 \Rightarrow \left(-\frac{ax}{by}\right)\left(-\frac{AX}{BY}\right) = -1$

$\Rightarrow aAx^2 = -bBy^2$ — (1)

(8)

Now $ax^2+by^2=1$ & $Ax^2+By^2=1$

$\Rightarrow ax^2+by^2 = Ax^2+By^2 \Rightarrow (a-A)x^2 = (B-b)y^2$

$\Rightarrow \frac{x^2}{y^2} = \frac{B-b}{a-A}$

from ① $\frac{x^2}{y^2} = \frac{-bB}{aA}$

$\Rightarrow \frac{B-b}{a-A} = -\frac{bB}{aA}$

$\Rightarrow \frac{B-b}{bB} = -\left(\frac{a-A}{aA}\right)$

$\Rightarrow \frac{1}{b}-\frac{1}{B} = \frac{1}{a}-\frac{1}{A}$

(31) $f'(x) = 3kx^2 - 18x + 9 = 3[kx^2-6x+3]$

$f'(x)>0 \Rightarrow kx^2-6x+3>0 \;\; \forall\; x\in R$

$\Rightarrow (-6)^2 - 4k\times 3 < 0$ & $k>0$

$\Rightarrow$ $k>3$ & $k>3$

$\Rightarrow \underline{k>3}$

(32) $\frac{dy}{dx} = f'(x) \Rightarrow f'(1) = \tan\pi/6 = \frac{1}{\sqrt3}$

$f'(2) = \tan\pi/3 = \sqrt3$

$f'(3) = \tan\pi/4 = 1$

$\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$

$\frac{[f'(x)]^2}{2}\Big|_2^3 + f'(x)\Big|_1^3 = \frac{[f'(3)]^2-[f'(2)]^2}{2} + f'(3)-f'$

(Q33) L. M.V.T. is not applicale on $|x|$ in $[-1,1]$ because $|x|$ is non differentiable at $x=0$

(34) $f(x) = 3ax^2 + 2bx + c$

let $g(x) = \int f(x)dx = \int (3ax^2 + 2bx + c)dx$

$= ax^3 + bx^2 + cx + d$

$g(0) = d$ , $g(2) = 8a + 4b + 2c + d = 2(4a+2b+c) + d$

$= d$

$g(0) = g(2)$ hence by rolle mean value theorem there exist at least one root of eq. $g'(x) = 0$

$\Rightarrow f(x) = 0$ in the interval of $(0, 2)$

(Q35) $f'(x) \geq 6$

$\int_2^4 f'(x)dx \geq \int_2^4 6dx$

$f(4) - f(2) \geq 6x\Big|_2^4 = 6\times 2$

$f(4) + 4 \geq 12$

$f(4) \geq 8$

(36) By rolle's mean value theorem $f(1) = f(2)$

$\int_1^2 f'(x)dx = f(2) - f(1) = 0$

(37) $f'(x) > 0 \Rightarrow f(x)$ is increasing

$g'(x) < 0 \Rightarrow g(x)$ is decreasing

$h(x) = f\{g(x)\} \Rightarrow h'(x) = \underbrace{f'\{g(x)\}}_{>0}\underbrace{g'(x)}_{<0}$

$\Rightarrow h'(x) < 0 \Rightarrow h(x)$ is decreasing function

since $x < x+1$

(a) $\Rightarrow fog(x) > fog(x+1)$ because $fog$ is decreasing fund

(b) $x > x-1$

$\Rightarrow fog(x) < fog(x-1)$ so (b) alternative incorrect.

(10)

Similarly verify other alternatives.

(38) Tangent is ~~to~~ parallel to x axis at $x = a$

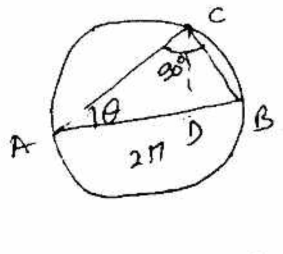
$\Rightarrow \quad f'(a) = 0$

$f'(a-h) > 0$ & $f'(a+h) < 0$ , $h > 0$

$\Rightarrow f'(x)$ changes sign changes from +ive to −ive

at $x = a \Rightarrow x = a$ is pt of maxima.

(39)



Area of ~~trance~~ triangle ABC

$= \frac{1}{2}(AB) \times (CD)$

AB is constant so area is maximum when CD is maximum

maximum when CD is maximum, C should on the ⊥ bisector

& for maximum CD

of AB ie $CD = r$

So For maximum area $\Delta ABC$ will be isosceles

For ~~perimeter~~ Perimeter.

let $\angle BAC = \theta$

$AB = 2r$

$AC = 2r\cos\theta$ , $BC = 2r\sin\theta$

$P = 2r(1 + \cos\theta + \sin\theta)$

$\frac{dP}{d\theta} = 2r(-\sin\theta + \cos\theta) = 0 \Rightarrow \tan\theta = 1$, $\theta = \pi/4$

$\frac{d^2P}{d\theta^2} = 2r(-\cos\theta - \sin\theta) = -ve \Rightarrow$ Perimeter is maximum

$\Rightarrow$ If P is maximum than $\Delta$ is isosceles

(40) $\frac{dy}{dx} = \frac{a(\cos\theta - \cos\theta + \theta\sin\theta)}{a(-\sin\theta + \sin\theta + \theta\cos\theta)} = \tan\theta$

Eqn. of normal $y - a(\sin\theta - \theta\cos\theta) = -\frac{1}{\tan\theta}\left(x - a(\cos\theta + \theta\sin\theta)\right)$

$\Rightarrow \sin\theta\left[y - a(\sin\theta - \theta\cos\theta)\right] + \cos\theta\left[x - a(\cos\theta + \theta\sin\theta)\right] = 0$ (ii)

Since slope of normal $= -\frac{1}{\tan\theta}$ which is not constant ie for different point $\theta$ is different

$\Rightarrow$ (a) is incorrect

(b) (0,0) does not satisfy the Eqn of normal $\Rightarrow$ (b) incorrect

(c) Distance from ~~origin~~ origin

$$d = \frac{\left| \sin\theta\left[0 - a(\sin\theta - \theta\cos\theta)\right] + \cos\theta\left[0 - a(\cos\theta + \theta\sin\theta)\right] \right|}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$$= \frac{a\left| -\sin^2\theta + \theta\sin\theta\cos\theta - \cos^2\theta - \theta\sin\theta\cos\theta \right|}{1}$$

$= a$ which is constant.

(41) $\frac{dy}{dx} = \frac{a\cos\theta}{a(-\sin\theta)} = -\frac{\cos\theta}{\sin\theta}$

$y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}\left[x - a(1+\cos\theta)\right]$

$\sin\theta\left[y - a\sin\theta\right] + \cos\theta\left[x - a(1+\cos\theta)\right] = 0$

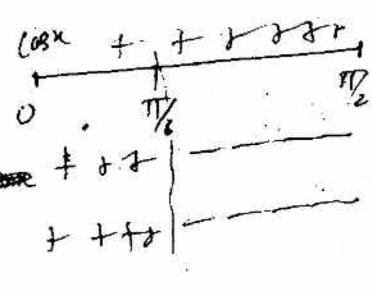
Check from alternatives.

(a,0) satisfies the Eqn.

(42) $f(x) = 1 + 2\sin x + 2\cos^2 x$

$f'(x) = 2\cos x - 4\cos x\sin x$

$= 4\cos x\left(\frac{1}{2} - \sin x\right)$




$\Rightarrow f(x)\uparrow \; x \in (0, \pi/6)$

$\downarrow \; x \in (\pi/6, \pi/2)$

f(x) first increases then decreases

$\Rightarrow x = \pi/6$ is point of maxima or greatest value

$f'(x) = 2x\, e^{-(x^2+1)^2} - 2x\, e^{-(x^2)^2}$

$= 2x\left[e^{-(x^2+1)^2} - e^{-x^4}\right]$

Now $(x^2+1)^2 > (x^2)^2$

$\Rightarrow -(x^2+1)^2 < -x^4$

$\Rightarrow e^{-(x^2+1)^2} < e^{-x^4} \Rightarrow e^{-(x^2+1)^2} - e^{-x^4} < 0, \forall x \in R$

So for $f'(x) > 0 \Rightarrow 2x\underbrace{\left[e^{-(x^2+1)^2} - e^{-x^4}\right]}_{-ive} > 0$

$\Rightarrow x < 0 \Rightarrow f(x)\uparrow \quad x \in (-\infty, 0)$

(44) let point be $\left(h, \frac{6h-5h^3}{3}\right)$

On diff. the curve $\Rightarrow 3\frac{dy}{dx} = 6 - 15x^2$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=h} = 2 - 5h^2$

Eqn. of Normal at $\left(h, \frac{6h-5h^3}{3}\right)$ is

$y - \left(\frac{6h-5h^3}{3}\right) = -\frac{1}{(2-5h^2)}(x-h)$

it passes through $(0,0)$

$\Rightarrow 0 - \left(\frac{6h-5h^3}{3}\right) = -\frac{1}{(2-5h^2)}(0-h)$

$\Rightarrow (6h-5h^3)(2-5h^2) = -3h$

$\Rightarrow h = 0$ or $(6-5h^2)(2-5h^2) = -3$

$\Rightarrow 25h^4 - 40h^2 + 15 = 0$

$\Rightarrow 25h^4 - 40h^2 + 3 = 0$

$\Rightarrow 5h^4 - 8h^2 + 3 = 0$

$\Rightarrow h^2 = 1$ or $-3/5$ (not possible)

$\Rightarrow h = \pm 1, 0$

(44) $h'(x) = f'(x) - 2f(x)f'(x) + 3[f(x)]^2 f'(x)$

$= f'(x)\left[1 - 2f(x) + 3(f(x))^2\right]$

$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$

(13) $= 3f'(x)\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$ = always +ve

$\Rightarrow$ ~~Sign~~ Sign of $h'(x)$ is similar to $f'(x)$ so whenever ~~...~~ $f(x)\uparrow$ $h(x)$ also $\uparrow$ & vice versa.

**45** let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

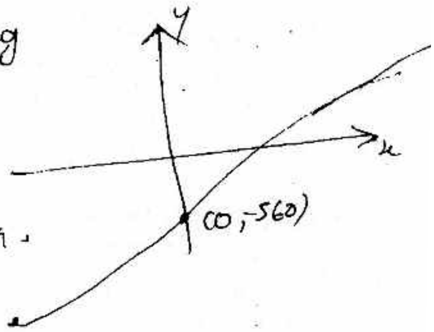
$f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall x \in R$

$\Rightarrow f(x)$ is always increasing

as $x \to -\infty \Rightarrow f(x) \to -\infty$

$x \to \infty \Rightarrow f(x) \to \infty$

so $f(x) = 0$ at only one value of $x \Rightarrow$ only 1 root of given Eq.




**46** Let $f(x) = e^x - 1 - x$

$f'(x) = e^x - 1 > 0, \ x \in (0, \infty)$

$\Rightarrow f(x) \uparrow$ in $(0, \infty)$

if $x > 0 \Rightarrow f(x) > f(0)$

$e^x - 1 - x > 0$

$\Rightarrow e^x > 1 + x, \quad x \in (0, \infty)$

Similarly verify others.

**47** $h'(x) = \underbrace{f'\{g(x)\}}_{+ive} \ \underbrace{g'(x)}_{-ive} \Rightarrow h'(x) < 0$

$\Rightarrow h(x)$ is decreasing

$\Rightarrow x > 0 \Rightarrow h(x) \le h(0)$

$\Rightarrow h(x) \le 0 \quad -(1)$

but in ~~ques~~ question it is given that $0 \le h(x) < \infty \quad -(2)$

from (1) & (2) $\Rightarrow h(x) = 0$

so $h(x) - h(1) = 0$

**48** $f'(-1) = 0$ & $f'(1/3) = 0$, $f'(x)$ is ~~a~~ quadratic polynomial

so $f'(x) = a(x+1)(x - 1/3) = a(x^2 + 2/3\, x - 1/3)$

$f(x) = \int f(x)dx = a\left(x^3/3 + \frac{1}{3}x^2 - x/3\right) + b$

$f(-1) = 0 \Rightarrow \frac{a}{3} \ldots + b = \frac{-2a}{3} + b = 0$ ...(1)

$\int_{-1}^{1} \ldots = \int_{-1}^{1} [\frac{a}{3}(x^3 + x^2 - x) + b] dx = \frac{14}{3}$

$\Rightarrow \frac{a}{9} \times 2 + 2b = \frac{14}{3}$

$\Rightarrow \frac{2a}{9} + 2b = \frac{14}{3}$ ...(2)

$\Rightarrow \frac{2a}{9} + \frac{4a}{3} = \frac{14}{3} \Rightarrow a = 3$ & $b = 2$

(49) let $f(x) = \ln(1+x) - x$ & $f(0) = 0$

$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$

for $x > 0 \Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing

$\Rightarrow x > 0 \Rightarrow f(x) < f(0)$

$\ln(1+x) - x < 0, \; x > 0$

(50) let $h(x) = e^{x-1} + x - 2$

$h'(x) = e^{x-1} + 1 > 0$

as $x \to -\infty \Rightarrow h(x) \to -\infty$

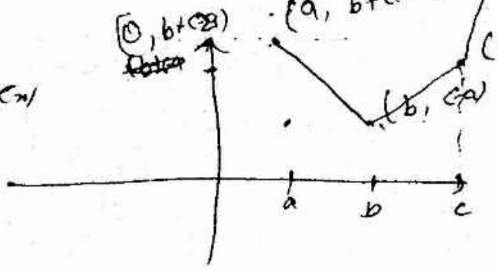
$x \to \infty \Rightarrow h(x) \to \infty$

so $h(x)$ ~~with~~ intersects the x axis at only one point $\Rightarrow$ only one point of intersection

(51)
$$f(x) = \begin{cases} a+b+c-3x, & x \le a \\ b+c-a-x, & a \le x \le b \\ c-a-b+x, & b \le x \le c \\ 3x-a-b-c, & x \ge c \end{cases} \Rightarrow f'(x) = \begin{cases} -3, & x < a \\ -1, & a \le x < b \\ 1, & b < x < c \\ 3, & x > c \end{cases}$$

$\Rightarrow f(x)$ decreases till b & after that increases.

Graph of $f(x)$




From graph minimum value of $f(x)$ is $c-a$ at $x = b$

(52) $f'(x) = -\sin x - 2a$

Since $-1 \le \sin x \le 1$

$\Rightarrow -1-2a \le -2a-\sin x \le 1-2a$

If $f'(x) > 0$, $\forall x \in R \Rightarrow$ minimum value of $f'(x)$ should be greater than zero

$\Rightarrow -1-2a > 0 \Rightarrow a < -\frac{1}{2}$

(53) Eqn of tangent

$y - 3 = \frac{3-(-2)}{0-5}(x-0)$

$y - 3 = -x \Rightarrow x+y-3=0$

curve $y = \frac{c}{x+1}$ & tangent is $y = 3-x$

solving them should give only one point of intersection

$\Rightarrow 3-x = \frac{c}{x+1} \Rightarrow (3-x)(x+1) = c$

$\Rightarrow -x^2+2x+3 = c$

$\Rightarrow x^2-2x+c-3 = 0$

For only one $x$ $D=0 \Rightarrow 4-4\times1(c-3)=0$

$\Rightarrow c-3=1$

$c = 4$

(54) let $f(x) = \int(4ax^3+3bx^2+2cx+d)dx$

$= ax^4+bx^3+cx^2+dx+e$

$f(0) = e$

$f(3) = 81a+27b+9c+3d+e = 3(27a+9b+3c+d)+e$

$= e$

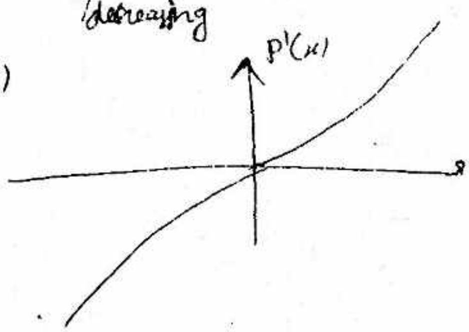
Hence by rolles mean value theorem there exist at least one root of Eq. $f'(x)=0$ in $(0,3)$

(55) $f(x) = 0$ have one roots $x = 0$ other ~~...~~ + the root
$\alpha$ given in the question.
So by perivious question there exist at least one root $(\alpha_1)$ of
Eq. $f'(x) = 0$ lying between $[0, \alpha]$
$f'(x)$ is polynomial of $4^{th}$ order polynomial so $f'(x) = 0$
have even no. of zero's. Since complex roots ~~...~~ are
~~...~~ always in pair with it's conjugate. ~~if it has~~
~~one root~~ if $f'(x) = 0$ have at ~~least~~ one real root
then there will also be second real root.
so $f'(x) = 0$, have at least two real roots.
Hence between any two real roots of $f'(x) = 0$ there
lies at least one root of $f''(x) = 0$. so
Ans (d) (all the three a, b, c are correct)

(56) $P'(x) = 4x^3 + 3ax^2 + 2bx + c$

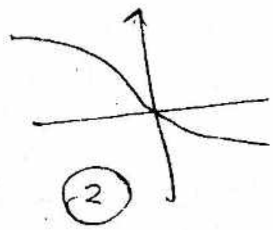
$P'(x) = 0$ have only one real roots $x = 0 \Rightarrow c = 0$ &
$P(x)$ is always increasing/decreasing function of $x$. (Because $P'(x)$ is Cubic polynomial

Graph of $P'(x)$



(1).

or



(2)

If $P'(x)$ is (1) $\Rightarrow$ $P(x) \uparrow$ in $(0, \infty)$
$\downarrow$ in $(-\infty, 0)$

$\Rightarrow$ $(-1, 0)$ $P(x)$ $\downarrow$ &
$(0, 1)$ $P(x)$ $\uparrow$

If (2)
$P(x)$ $\uparrow (-\infty, 0)$
$\downarrow (0, \infty)$

$P(x) \uparrow (-1, 0)$ & $(0, 1)$

Since $P(-1) < P(1)$ $\Rightarrow$ Second case is not possible.
$\Rightarrow x = 0$ is pt. of minima
~~If first~~ . from (1)
of $P(x)$ & maxima may be $P(1)$
or $P(-1)$.
But $P(1) > P(-1)$ $\Rightarrow$ maxima of $P(x)$ is $P(1)$

So Ans (a)

(57) $f'(x) = x^2 + 2x + 2 > 0$ for $x \in [2, 4]$.
so $f_{max} = f(4) = \int_0^4 (t^2 + 2t + 2)\,dt$.
& $f_{min} = f(2) = \int_0^2 (t^2 + 2t + 2)\,dt$.

(58) $F'(x) = \dfrac{\sin x}{x} = 0 \Rightarrow x = n\pi$
Since $x > 0$ $\Rightarrow$ $x = \pi$ $\sin x$ changes sign from +ive to −ive $\Rightarrow$ point of local maxima

at $x = 2\pi$ Singn of $\sin x$ changes from $-ve$ to $+ve$
$\Rightarrow$ point of minima

so $x = (2n+1)\pi$ are point of local maxima
& $x = 2n\pi$ are point of local minima

(59) $f(x) = \dfrac{1}{e^x + 2e^{-x}} =$

$$\frac{e^x + 2e^{-x}}{2} \geq \sqrt{e^x \cdot 2e^{-x}} \quad (A.M. \geq G.M.)$$

$$\Rightarrow e^x + 2e^{-x} \geq 2\sqrt{2}$$

$$\therefore \text{so } \frac{1}{e^x + 2e^{-x}} \leq \frac{1}{2\sqrt{2}}$$

so range of $f(x)$ is $\left(0, \frac{1}{2\sqrt{2}}\right]$

Since $\frac{1}{3} \in \left(0, \frac{1}{2\sqrt{2}}\right] \Rightarrow$ there exist some value of $c$ for which $f(c) = \frac{1}{3}$

so (d) is correct.

(60) $f(x)$ is continuous but not differentiable at $x = 0, \pi/2$

$f'(0-) = 4$ & $f'(0+) = -1$ so $x = 0$ is point of local maxima.

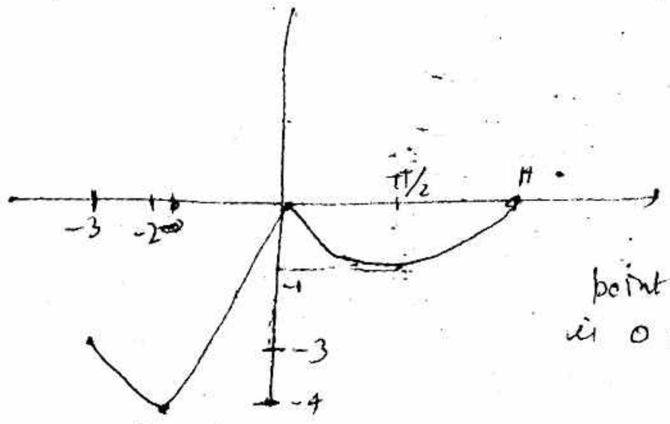
$f'(\pi/2 -) = 0$ & left of $\pi/2$ it is $-ve$ $(-\cos x)$

$f'(\pi/2 +) = 1$ so $x = \pi/2$ is point of local minima.

$$f'(x) = \begin{cases} 2x+4, & -3 \leq x \leq 0 \\ -\cos x, & 0 < x < \pi/2 \\ \sin x, & \pi/2 < x \leq \pi \end{cases}$$

$2x + 4 = 0$
$\Rightarrow x = -2$ is point of local minima

Graph of f(x)




point of global maxima is 0 or π

point of Global minima.

(61) Eqn. of tangent $\frac{x}{3\sqrt{3}}\cos\theta + y\sin\theta = 1$

$$x_{int} = \frac{3\sqrt{3}}{\cos\theta}, \quad y_{int} = \frac{1}{\sin\theta}$$

$$S = \frac{3\sqrt{3}}{\cos\theta} + \frac{1}{\sin\theta}, \quad \frac{dS}{d\theta} = -\frac{3\sqrt{3}}{\cos^2\theta}(-\sin\theta) - \frac{1}{\sin^2\theta}\cos\theta = 0$$

$$\Rightarrow \frac{3\sqrt{3}}{\cos^2\theta}\sin\theta = \frac{1}{\sin^2\theta}\cos\theta$$

$$\Rightarrow \tan^3\theta = \frac{1}{3\sqrt{3}}$$

$$\Rightarrow \theta = \pi/6$$ (check it is point of minima

(62) $f'(x) = 3x^2 + 2bx + c$

$$D = 4b^2 - 4\times 3c = 4(b^2 - 3c)$$
$$= 4(b^2 - c) - 4\times 2c$$
$$= (-ve) + (-ve) = -ve$$

so $D < 0$ & coff of $x^2 = 3 > 0$

$\Rightarrow f'(x) > 0, \ \forall\ x \in R.$

$\Rightarrow f(x)$ is increasing in $(-\infty, \infty)$

(63)
$$f(x) = \begin{cases} x^{\alpha}\ln x, & 0<x\le 1 \\ 0, & x=0 \end{cases}$$

for continuity of $f(x)$ at $x=0$

$\lim_{x\to 0+} x^{\alpha}\ln x = 0 \Rightarrow \alpha>0$ — ①

$f(x)$ is diff. $x\in(0,1)$ — ②

from ① & ② rolle's mean value theorem is applicable to $f(x)$.

(64) let $P(x) = px^2+qx+r$

$P(0)=0 \Rightarrow r=0$ — ①

$P(1)=1 \Rightarrow p+q=1$ — ②

let $q=a$ & $p=1-a$ satisfying ① & ②

$\Rightarrow P(x) = ax+(1-a)x^2 > 0, \forall x\in[0,1]$

$P'(x) = a+2(1-a)x$ & $P'(1) = a+2-2a = 2-a$

$P'(0)=a$

for $P'(x)>0 \Rightarrow a>0$ & $2-a>0$

$\Rightarrow \underline{0<a<2}$

(65) $f(x) = \dfrac{x^2-1}{x^2+1}$, $\lim_{x\to\pm\infty} f(x)=1$ so $f(x)$ is bounded.

$f(x) = 1-\dfrac{2}{x^2+1}$

$f'(x) = \dfrac{2\times 2x}{(x^2+1)^2} \Rightarrow f(x)\uparrow x\in(0,\infty)$

$\downarrow x\in(-\infty,0)$

$\Rightarrow x=0$ is point of minima & end values are not attained because they are obtained at $x=$

So minimum is 1 & maximum is not attained even $f(x)$ is bounded.

66 $f'(x) = \cos\frac{1}{x} + x\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$= \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$

(b). $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \cos\frac{1}{x} + \frac{1}{x}\left(\sin\frac{1}{x}\right)$

$\cos 0 + \cdot 0 = 1$

~~(d) If $x>1 \Rightarrow \frac{1}{x} \in (0,1)$~~

~~so $\cos\frac{1}{x} > 0$ + $\sin\frac{1}{x} > 0$~~

(d). $f''(x) = -\left(\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - \frac{1}{x^2}\sin\frac{1}{x} + \frac{1}{x}\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$= -\frac{1}{x^3}\cos\frac{1}{x}$

If $x>1 \Rightarrow 0 < \frac{1}{x} < 1$ so $\cos\frac{1}{x} > 0$

hence $f''(x) < 0$

$\Rightarrow f'(x)$ is $\downarrow$ $x \in (1,\infty)$

(a) or (c) Using Lagrange's mean value theorem in $f(x)$ from $x$ to $(x+2)$

$\Rightarrow \frac{f(x+2) - f(x)}{x+2-x} = f'(c)$ — (1), $c \in (x, x+2)$

~~f(x)~~ $f'(x)$ is ~~~~ $\downarrow$ & $\lim_{x\to\infty} f'(x) = 1$

so $f'(x) < 1$

$\Rightarrow f'(c) < 1$ — (2)

Using (1) & (2)

$\Rightarrow \frac{f(x+2) - f(x)}{2} < 1 \Rightarrow f(x+2) - f(x) < 2$

67

~~f(x) decreases~~
$$f'(x) = \begin{cases} -2, & x<-1 \\ 2, & x>-1 \end{cases} \Rightarrow f(x)\downarrow \text{ for } x<-1 \;\&\; f(x)\uparrow \text{ for } x>-1$$

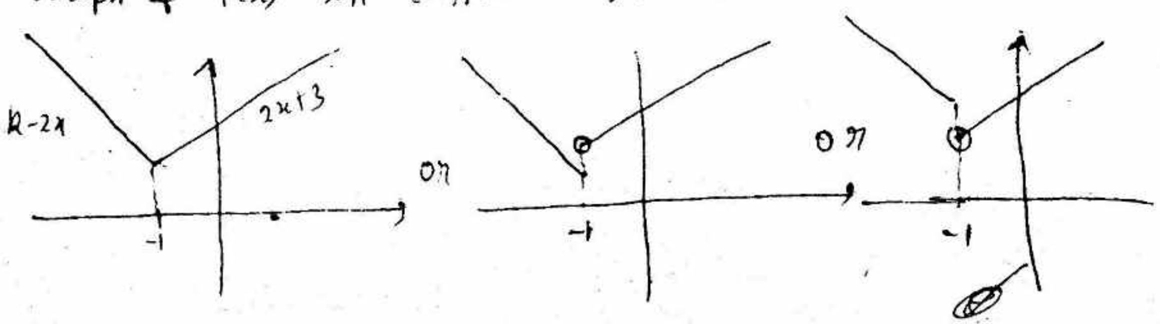
f(x) has minimum value at $x=-1$ only if

$f(-1) \le \lim_{x\to -1^+} f(x)$

$k+2 \le -2+3 \Rightarrow k \le -1$

(D)

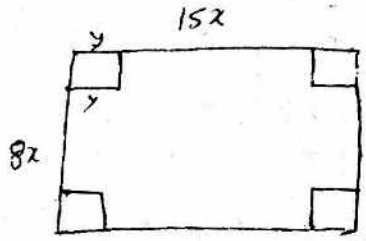
Graph of f(x) in diff ... :




Minimum at $x=-1$ Minimum at $x=-1$ not minimum at $x=-1$

68

let sides be $8x$, $15x$ &
if square of y length is cut




$V = (8x-2y)(15x-2y)y$

$V = (120x^2 - 46xy + 4y^2)y$

$= 4y^3 - 46xy^2 + 120x^2y$

$4y^2 = 100$
$y = 5$

V is maximum when $y=5$

$\frac{dV}{dy} = 12y^2 - 92xy + 120x^2$

$\left.\frac{dV}{dy}\right|_{y=5} = 0 \Rightarrow 12 \times 25 - 92 \times 5x + 120x^2 = 0$

$5\left(60 - 92x + 24x^2\right] = 0$

$4\left(6x^2 - 23x + 15\right) = 0$

$6x^2 - 18x - 5x + 15 = 0$

$6x(x-3) - 5(x-3) = 0$

$(6x-5)(x-3) = 0$

$x = 3$ or $5/6$

$\frac{d^2V}{dy^2} = 24y - 92x$

$\left.\frac{d^2V}{dy^2}\right|_{y=5} = 120 - 92x$ when $x = 5/6$ $\frac{d^2V}{dy^2} > 0 \Rightarrow$ not maximum

(25) when $x = 3$ $\frac{d^2V}{dy^2} < 0 \Rightarrow$ maximum

So $x = 3$

So sides are 24 & 45

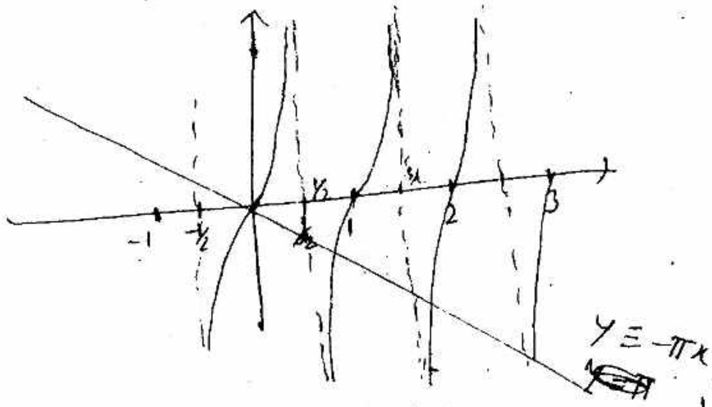
(69) $f(x)f'(x) < 0$ $\Rightarrow [f(x)]^2$ is decreasing function

$\Rightarrow \frac{1}{2}\frac{d}{dx}[f(x)]^2 < 0$ $\Rightarrow |f(x)|$ is decreasing function

(70) $f'(x) = \sin\pi x + x\pi\cos\pi x = 0$

$\Rightarrow \sin\pi x = -x\pi\cos\pi x$ Now solve graphically.

$\Rightarrow \tan\pi x = -x\pi$

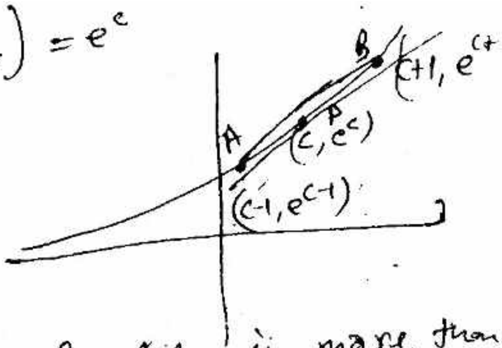



Clearly ~~curves~~ both curve meet between any two consecutive ~~integers~~ integers. so (c) is correct.

Also solution lies between $(1/2, 1)$, $(3/2, 2)$, $(5/2, 3)$ and so on

$\Rightarrow$ unique solution also lies in $(n+1/2, n+1) \Rightarrow$ (b) is correct

(71) Slope of tangent at P $\left(\frac{dy}{dx}\Big|_{x=c}\right) = e^c$

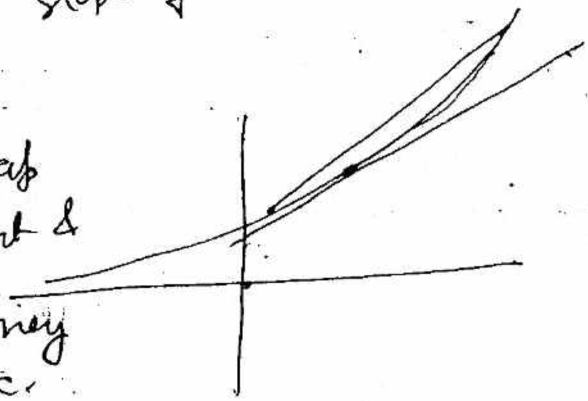



Slope of AB line

$$\frac{e^{c+1} - e^{c-1}}{(c+1)-(c-1)} = \frac{e^c\left(e - \frac{1}{e}\right)}{2}$$

$\frac{e - \frac{1}{e}}{2} > 1$ so slope of AB is more than the tangent at P

⇒ on the left of P gap between tangent & AB decreases so they intersect at left of c.



(72) let $f(x) = ax^3 + bx^2 + cx + d$

$f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ — (1)

$f(1) = -1 \Rightarrow a + b + c + d = -1$ — (2)

$f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$ — (3)

$f''(0) = 0 \Rightarrow 2b = 0$ — (4)

Find a, b, c, d & verify the alternatives.

(73) $e^{-\pi/2} < \theta < \pi/2$

$-\pi/2 < \ln\theta < \ln \pi/2$ & $\ln \pi/2 < 1$

$\ln\theta$ belongs to first & fourth quadrant

$\Rightarrow \cos(\ln\theta) > 0$

$\ln\cos\theta < 0$ because $\cos\theta \in (0,1)$

hence $\cos(\ln\theta) > \ln(\cos\theta)$

**34** $\frac{dy}{dx} = 4x - \frac{1}{x}$ , $x \neq 0$ (3)

$= \frac{4x^2-1}{x} = \frac{4(x-\frac{1}{2})(x+\frac{1}{2})}{x}$

$\frac{dy}{dx}$: $-$ , $+$ , $-$ , $+$ ; $x$: $-\frac{1}{2}$, $0$, $\frac{1}{2}$

$y \uparrow \quad x \in (-\frac{1}{2}, 0) \cup (\frac{1}{2}, \infty)$

$y \downarrow \quad x \in (-\infty, -\frac{1}{2}) \cup (0, \frac{1}{2})$

**35** $\log_a x + \log_x a = \log_a x + \frac{1}{\log_a x}$

sum of two numbers which are reciprocal of each other is always $\geq 2$ or $\leq -2$

$0 < a < x$ no it can't be said that $\log_a x$ is greater than zero or less than zero.

If $\log_a x > 0$ then minimum value will be 2 ~~otherwise~~

If $\log_a x < 0 \Rightarrow$ maximum value will be $-2$

So given statement is false.

**36** $f(x) = \begin{cases} -1 , & -2 \leq x \leq 0 \\ x-1 , & 0 < x \leq 2 \end{cases}$ $\quad g(x) = |x|$

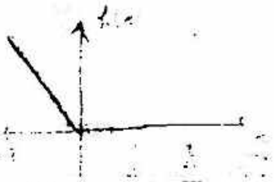
$fog = f\{g(x)\} = f(|x|) = |x| - 1$, because $|x| \geq 0$

$gof = g\{f(x)\} = \begin{cases} g(-1) , & -2 \leq x \leq 0 \\ g(x-1) , & 0 < x \leq 2 \end{cases}$

$= \begin{cases} 1 , & -2 \leq x \leq 0 \\ |x-1| , & 0 < x \leq 2 \end{cases}$

$h(x) = \begin{cases} |x|-1-1 & -2 \leq x \leq 0 \\ |x-1| + |x| - 1 & 0 \leq x \leq 2 \end{cases} = \begin{cases} -x , & -2 \leq x \leq 0 \\ 0 , & 0 < x \leq 1 \\ 2x-2 , & 1 \leq x \leq 2 \end{cases}$

Graph of $h(x)$



so range is $[0,2]$

(77) (a) Graph is symmetric about y axis $\Rightarrow f'(x)$ is even fuction.

(b) $f'(x)$ is even & $f(0) = 0 \Rightarrow f(x)$ is odd function

(c) $\int_{-a}^{a} f(x)dx = 0$ because $f(x)$ is odd

(d) $f'(x)$ is always $\leq 0$ & zero at $x = 0$ so $x = 0$ is point of inflection of $f(x)$

(e) $f(x)$ is always decreasing because $f'(x) \leq 0$ in $-a \leq x \leq a$

(78) 1. $g'(x) = \left[\frac{2(x-1)}{x+1} - \ln x\right] f(x)$

$f(x) > 0$

let $h(x) = \frac{2(x-1)}{x+1} - \ln x$

$h'(x) = \frac{4}{(1+x)^2} - \frac{1}{x} = \frac{4x - (1+x)^2}{(1+x)^2}$

$= \frac{-(1-x)^2}{(1+x)^2} < 0$

$\Rightarrow h(x)$ is $\downarrow$

$h(1) = 0$ so if $x > 1$

$\Rightarrow h(x) < h(1)$

$\Rightarrow h(x) < 0$

So $g'(x) < 0$, $\forall x > 1$

$\Rightarrow g(x)$ is $\downarrow$ $x \in (1, \infty)$

2. For p. $(1-x)^2 \sin^2 x + x^2 + 2x = 2(1+x^2)$

$\Rightarrow (1-x)^2 \sin^2 x = x^2 - 2x + 2$

$\Rightarrow \sin^2 x = \frac{x^2 - 2x + 2}{(1-x)^2} = 1 + \frac{1}{(1-x)^2}$

Clearly R.H.S. $> 1$ L.H.S. $\leq 1$

$\Rightarrow$ both sides cannot be equal $\Rightarrow$ P is statement

Q. $2(1-x)^2 \sin^2 x + 2x^2 + 1 = 2x + 2x^2$

$\Rightarrow \cancel{2(1-x)^2 \sin^2 x = x^2+2x-1}$

$\Rightarrow \cancel{\sin^2 x = \frac{x^2+2x-1}{2(1-x)^2}} =$

$\Rightarrow 2(1-x)^2 \sin^2 x = 2x-1$

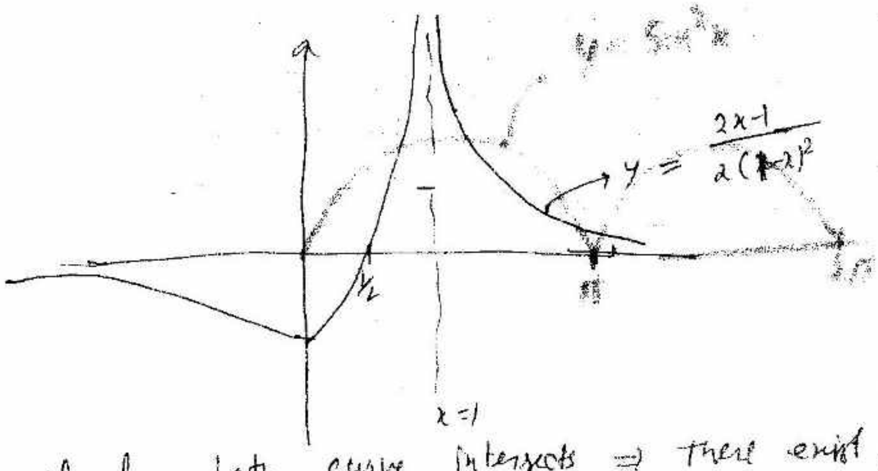
$\Rightarrow \sin^2 x = \frac{2x-1}{2(1-x)^2} = \cancel{-\frac{1}{1-x} + \frac{1}{2(1-x)^2}}$

~~let f(x)~~ To identify both sides are equal for some $x$ we plot the graph of both the side & see ~~weather~~ they intersect or not

$y = \frac{2x-1}{2(1-x)^2} \Rightarrow \frac{dy}{dx} = \frac{x(1-x)}{(1-x)^4}$

For curve $x=1$ is vertical asymptote.

$\frac{dy}{dx}$ $- - -$ $+ +$ $- -$ ; $x$: $0$, $1$




Clearly both curve intersects $\Rightarrow$ there exist some $x$.

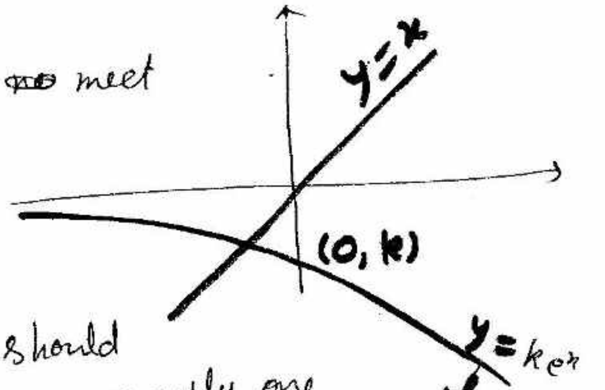
Alt. Let $h(x) = 2(1-x)^2 \sin^2 x - 2x + 1$

$h(0) = 1$
$h(1) = -1$

so $h(0)$ & $h(1)$ are opposite in sign & $h(x)$ is continuous function $\Rightarrow h(x) = 0$ for at least one value of $x$.

Q 88

(a) From graph curve meet at one point.




(b) ~~ke^x~~ curve

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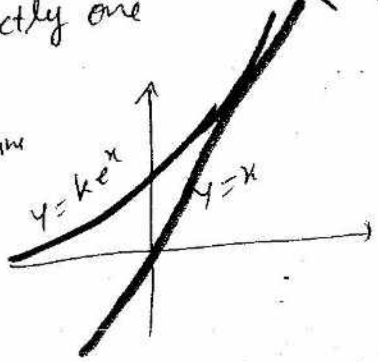
$y = ke^x$ & $y = x$ should touch each other for exactly one root. $(k \geq 0)$




let point of contact be $(\alpha, \alpha)$ on line $y = x$

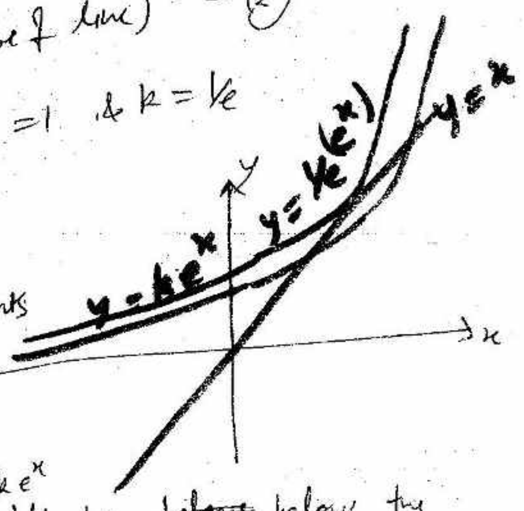
$(\alpha, \alpha)$ lies on $y = ke^x$

$\Rightarrow \alpha = ke^{\alpha}$ — (1)

$\frac{dy}{dx} = ke^x \Rightarrow \frac{dy}{dx}\Big|_{x=\alpha} = ke^{\alpha}$

$\Rightarrow ke^{\alpha} = 1$ (slope of line) — (2)

From (1) & (2) $\Rightarrow \alpha = 1$ & $k = 1/e$

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(c) For two distinct roots curve $y = ke^x$ & $y = x$ should intersect at two points

from (b) if $k = 1/e$ curve touches each other

if $k < 1/e$ then curve $y = ke^x$ will be ~~below~~ below the curve $y = ke^x$ (when $k = 1/e$) so it intersects & after that lies below the line $y = x$.

but as $x \to \infty$, $ke^x$ will be bigger than $x$ so again curve $y = ke^x$ will cross the curve $y = x$ & become above the line $y = x$ so $k \in (0, \frac{1}{e})$

(77)

Q91 $f(x) = \frac{x^2-ax+1}{x^2+ax+1} = \frac{x^2+1+ax-2ax}{x^2+ax+1} = 1-\left(\frac{2ax}{x^2+ax+1}\right)$

$f'(x) = \frac{-2a\left[-x(2x+a)+(x^2+ax+1)\times 1\right]}{(x^2+ax+1)^2}$

$= \frac{+2a(x^2-1)}{(x^2+ax+1)^2}$ — (1)

$f''(x) = \frac{+2a\left[(x^2+ax+1)^2\times 2x-(x^2-1)\times 2(x^2+ax+1)(2x+a)\right]}{(x^2+ax+1)^4}$

$= \frac{4a(x^2+ax+1)\left[x(x^2+ax+1)-(x^2-1)(2x+a)\right]}{(x^2+ax+1)^4}$

$= \frac{+4a(-x^3+3x+a)}{(x^2+ax+1)^3}$ — (2)

$f''(1) = \frac{+4a(a+2)}{(a+2)^3} = +\frac{4a}{(a+2)^2}$

$f''(-1) = +\frac{4a(a-2)}{(2-a)^3} = -\frac{4a}{(a-2)^2}$

so $(a+2)^2 f''(1) + (a-2)^2 f''(-1) = 0$

Q92 From Eq. (1) $a>0$

$x^2-1>0$ when $x>1$ or $x<-1$

& $x^2-1<0$ when $-1<x<1$

so $f'(x)<0$ when $-1<x<1$

$>0$ when $x>1$ or $x<-1$

so $f(x)$ ↑ $x \in (-\infty,-1)$ & $(1,\infty)$

$f(x)$ ↓ $x \in (-1,1)$

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$$g'(x) = \frac{f'(e^x)}{1+(e^x)^2}$$

$$= \frac{2a\left[e^{2x}-1\right]}{\left(e^{2x}+a e^x+1\right)^2\left[1+e^{2x}\right]}$$

$g'(x) > 0 \Rightarrow x > 0$ & $g'(x) < 0 \Rightarrow x < 0$