JEE (ADVANCE) SOLUTIONS – 2015 – CODE '8'
MATHEMATICS
PAPER - 2

41. $$\frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11}$$

$14a + 42d = 12a + 60\,d$

$2a = 18d \Rightarrow a = 9d$

$a_7 = a + 6d = 15d$

$130 < 15d < 140$

$26 < 3d < 28$

$8.67 < d < 9.33 \Rightarrow d = 9$

42. $9 = 9$ ...(1)

$= 8 + 1$ ...(2)

$= 7 + 2$ ...(3)

$= 6 + 3$ ...(4)

$= 5 + 4$ ... (5)

$= 1 + 2 + 6$ ...(6)

$= 1 + 3 + 5$ ...(7)

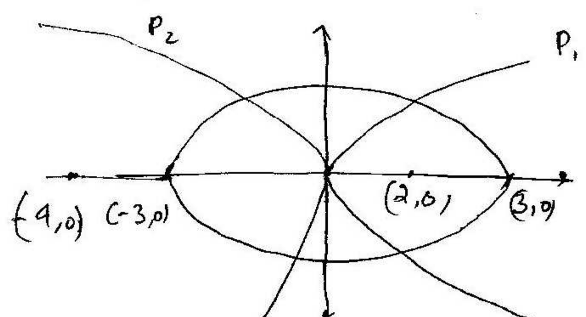
$= 2 + 3 + 4$ ...(8)

In product whenever use take following power every time coeff obtained is so coeff. of $x^9 = 8$

43. $5 = 9(1 - e^2) \Rightarrow e = \frac{2}{3}$

Focus $= \left(\pm 3 \times \frac{2}{3},\ 0\right) = (\pm 2, 0)$

$(f_1, 0) = (2, 0)$ and $(f_2, d) = (-2, 0)$



P_1 : $y^2 = 4 \times 2x$

$\Rightarrow y^2 = 8x$

P_2 : $y^2 = -4 \times 4x$

$y^2 = -16x$

Tangent to P_1

$t_1 y = x + 2t^2$

Passes through (–4, 0)

$\Rightarrow \quad 0 = -4 + 2t^2$

$\Rightarrow \quad t^2 = 2 \quad \Rightarrow \quad k_1 = \pm\sqrt{2}$

Slope $= \frac{1}{t} \quad \Rightarrow \quad m_1 = \pm \frac{1}{\sqrt{2}}$

Targent to P_2

$t_1 y = -x + 4t_1^2$ passes through (2, 0)

$0 = -2 + 4\,t_1^2$

$2t_1^2 = 1 \quad \Rightarrow \quad t_1 = \pm \frac{1}{\sqrt{2}}$

$m_2 = \frac{1}{t_2} = \pm\sqrt{2}$

$m_1^2 + \frac{1}{m_2^2} = (\sqrt{2}|^2 + (\sqrt{2}|^2 = \boxed{4}$

44. $\lim_{a\to 0} \frac{e^{\cos(a^n)} - e}{a^m} = -\frac{e}{2}$

$\lim_{a\to 0} \frac{e\left[e^{\cos a^n - 1} - 1\right]}{a^m\left[\cos a^n - 1\right]} = 1$

$\lim_{a\to 0} \frac{e\left[\cos a^n - 1\right]}{a^m} = \lim_{a\to 0} \frac{e\left[-2\sin^2 \frac{a^n}{2}\right]}{a^m \left(\frac{a^n}{2}\right|^2} \left(\frac{a^n}{2}\right|^2$

$= \lim_{a\to 0} -\frac{e}{2}\frac{a^{2n}}{a^m}$

$= \lim_{a\to 0} -\frac{e}{2} a^{2n-m}$

If above limit is $-\frac{e}{2}$ then $2n - m = 0 \Rightarrow 2n = m$

45. Let $9x + 3\tan^{-1} x = t$

$\left(9 + \frac{3}{1+x^2}\right)dx = dt$

$\frac{1^2 + 9x^2}{1+x^2}dx = dt$

$\int e^{9x + 3\tan^{-1}x}\left(\frac{12+9x^2}{1+x^2}\right)dx = \int e^t\, dt = e^t = e^{9x+3\tan^{-1}x}$

$\alpha = e^{9x+3\tan^{-1}x}\Big|_0^1 = e^{9+3\frac{\pi}{4}} - 1$

$\alpha + 1 = e^{9+3\frac{\pi}{4}}$

$\log_e(\alpha+1) = 9 + 3\frac{\pi}{4} \Rightarrow \log_e(\alpha+1) - 3\frac{\pi}{4} = 9$

46. f(0) = 0 (odd function is the only zero of f(x)

$$\lim_{x\to 1}\frac{\int_{-1}^{x} f(l)\,dt}{\int_{-1}^{x} t\left|f(f(t))\right| dt}$$

Applying L.H. rule

$$\lim_{x\to 1}\frac{f(x)}{x\left|f\{f(t)\}\right|}=\frac{1}{14}$$

$$\frac{f(1)}{f(f(x))}=\frac{1}{14}$$

$$\frac{\frac{1}{2}}{f\left(\frac{1}{2}\right)}=\frac{1}{14} \quad\Rightarrow\quad f\left(\frac{1}{2}\right)=7$$

48. $$\frac{\sum_{k=1}^{12}\left|e^{i(kH)\pi/7}-e^{ik\pi/7}\right|}{\sum_{k=1}^{3}\left|e^{i(4k-1)\pi/7}-e^{i(4k-2)\pi/7}\right|}=\frac{\sum_{k=1}^{12}\left|e^{i\frac{k\pi}{7}}\left(e^{i\pi/7}-1\right)\right|}{\sum_{k=1}^{3}\left|e^{i(4k-2)\pi/7}\left(e^{i\pi/7}-1\right)\right|}$$

$$=\frac{\sum_{k=1}^{12}\left|e^{ik\pi/7}\right|\left|e^{i\pi/7}-1\right|}{\sum_{k=1}^{3}\left|e^{i(4k-2)\pi/7}\right|\left|e^{i\pi/7}-1\right|}$$

$$=\frac{\sum_{k=1}^{12}\left|e^{i\pi/7}-1\right|}{\sum_{k=1}^{3}\left|e^{i\pi/7}-1\right|}$$

$$=\frac{12\left|e^{i\pi/7}-1\right|}{3\left|e^{i\pi/7}-1\right|}=4$$

$$\left(\left|e^{ik\frac{\pi}{7}}\right|=1\right)$$

49.

x	−1	0	2
(f−3g) (x)	3	3	3

Since $(f-3g)(-1)=(f-3g)(0)=(f-3g)(2)$

$\Rightarrow (f-3g)^{1}=0$ at least once in a interval of (−1, 0) and (0, 2) …(1)

Since $(f-3g)^{11}$ is never zero in (−1, 0) and (0, 2) so $(f-3g)^{1}$ is at the most once zero in (−1, 00 and (0, 2) … (2)

Combining (1) and (2) $\Rightarrow$ $(f-3g)^{1}=0$ exactly once in (−1, 0) and (0, 2)

50. $f(x)=7\tan^6 x\,(\tan^2 x=1)-3\tan^2 x\,(\tan^2 x+1)$

$=7\tan^6 x\sec^2 x-3\tan^2 x\sec^2 x$

$\int_0^{\pi/4} x\,f(x)\,dx=\int_0^{\pi/4} 3x\tan^2 x\sec^2 x\,dx$ integrating by parts taking x as first function

$$=x\tan^7\Big|_0^{\pi/4}-\int_0^{\pi/4}\tan^7 x\,dx-x\tan^3 x\Big|_0^{\pi/4}+\int_0^{\pi/4}\tan^3 x\,dx$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \tan^7 x\, dx - \frac{\pi}{4} + \int_0^{\pi/4} \tan^3 x\, dx$$

$$= \int_0^{\pi/4} \left(\tan^3 x - \tan^7 x\right) dx$$

$$= \int_0^{\pi/4} \tan^3 x \left(1 - \tan^2 x\right)\left(1 + \tan^2 x\right) dx$$

$$= \int_0^{\pi/4} \tan^3 x \left(1 - \tan^2 x\right) \sec^2 x \; dx$$

$$= \int_0^1 \left(t^3 - t^5\right) dt = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$$\int_0^{\pi/4} f(x)\, dx = \int_0^{\pi/4} 7 \tan^6 x \sec^2 x\, dx - 3 \int_0^{\pi/4} \tan^2 x \sec^2 x\, dx$$

$$= \tan^7 x\Big|_0^{\pi/4} - \tan^3 x\Big|_0^{\pi/4}$$

$$= 1 - 1 = 0$$

So (A) and (B) are correct

51. $f'(x) = \dfrac{192\, x^3}{2 + \sin^4 \pi x}$

Integrating both side $\int_{1/2}^{x} f'(x)\, dx = \int_{1/2}^{x} \dfrac{192\; x^3}{2 + \sin^4 \pi x} dx$ in $x \in \left[\dfrac{1}{2}, 1\right]$

$$0 \le \sin^4 \pi x \le 1$$

$$\le 2 + \sin^4 \pi x \le 3$$

$$\Rightarrow \frac{192\, x^3}{3} \le \frac{192\, x^3}{2 + \sin^4 \pi x} \le \frac{192\, x^3}{2}$$

$$\Rightarrow 64\, x^3 \le f'(x) \le 96\, x^3$$

$$\Rightarrow \int_{1/2}^{x} 64\, x^3 dx \le \int_{1/2}^{x} f'(x)\, dx \le \int_{1/2}^{x} 96\, x^3\, dx$$

$$\Rightarrow 16\left(x^4 - \frac{1}{16}\right) \le f(x) \le 24\left(x^4 - \frac{1}{16}\right)$$

Now integrating with limit ½ to 1

$$\Rightarrow \int_{1/2}^{1} 16\left(x^4 - \frac{1}{16}\right) dx \le \int_{1/2}^{1} f(x) dx \le \int_{1/2}^{1} 24\left(x^4 - \frac{1}{16}\right) dx$$

$$\Rightarrow \frac{16}{5}\left(1 - \frac{1}{32}\right) - \left(\frac{1}{2}\right) \le \int_{1/2}^{1} f(x) dx \le \frac{24}{5}\left(1 - \frac{1}{32}\right) - \frac{3}{2} \times \frac{1}{2}$$

$$\Rightarrow \frac{\cancel{16}}{5} \times \frac{31}{\cancel{32}_{2}} - \frac{1}{2} \le \int_{1/2}^{1} f(x)\, dx \le \frac{\overset{3}{\cancel{24}}}{5} \times \frac{31}{\cancel{32}_{4}} - \frac{3}{4}$$

$$\Rightarrow \frac{31}{10} - \frac{5}{10} \le \int_{1/2}^{1} f(x) dx \le \frac{93}{20} - \frac{15}{20}$$

$$\Rightarrow \frac{26}{10} \le \int_{1/2}^{1} f(x) dx \le \frac{78}{20}$$

From alter-native only (D) satisfies.

52. $D > 0$

$\Rightarrow 1 - 4a^2 > 0 \Rightarrow a^2 < \dfrac{1}{4} \Rightarrow -\dfrac{1}{2} < a < \dfrac{1}{2}$ ...(1)

$|x_1 - x_2| < 1$

$$\left|\frac{\sqrt{D}}{a}\right| < 1$$

$$\left|\frac{\sqrt{1-4a^2}}{a}\right| < 1 \Rightarrow \frac{1-4a^2}{a^2} < 1$$

$$\Rightarrow \frac{1-4a^2-a^2}{a^2} < 0$$

$$\Rightarrow 1-5a^2 < 0$$

$$\Rightarrow 5a^2 > 1$$

$$\Rightarrow a^2 > \frac{1}{5}$$

$$\Rightarrow a > \frac{1}{\sqrt{5}} \text{ and } a < -\frac{1}{5} \quad \ldots(2)$$

Common of (1) and (2) is $a \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

So A and D are correct

53. $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$

Since $\frac{1}{2} < \frac{6}{11}$

$\Rightarrow \sin^{-1}\frac{1}{2} < \sin^{-1}\frac{6}{11}$

$\Rightarrow \frac{\pi}{6} < \frac{\alpha}{3} \Rightarrow \alpha > \frac{\pi}{2}$

$\beta = 3\cos^{-1}\frac{4}{9}$

$\frac{1}{2} > \frac{4}{9}$

$\cos^{-1}\frac{1}{2} < \cos^{-1}\frac{4}{9}$ ($\cos^{-1}x$ is decreasing curve)

$\frac{\pi}{3} < \frac{\beta}{3}$

$\Rightarrow \beta > \pi$

So α lies in 2^nd quadrant and β in third, so $\cos\alpha < 0, \sin\beta < 0$

$\alpha+\beta$ will lie in 4^{th} so $\cos(\alpha+\beta) > 0$

54. P is mid point of Q R

P is intersection of line and circle

$x + y = 3$ and $x^2 + (y-1)^2 = 2$

$\Rightarrow (3-y)^2 + (y-1)^2 = 2$

$\Rightarrow 2y^2 - 8y + 8 = 0 \Rightarrow y^2 - 4y + 4 = 0 \Rightarrow (y-2)^2 = 0 \Rightarrow y = 2$

$y = 2 \Rightarrow x = 1$

so p is (1, 2)

slope of line = – 1

Parametric form of line

$$\frac{x-1}{-\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm\frac{2\sqrt{2}}{3}$$

Taking + sign $\Rightarrow$ $x = 1 - \frac{2}{3}$ and $y = 2 + \frac{2}{3}$

$$Q \Rightarrow (x, y) = \left(\frac{1}{3}, \frac{8}{3}\right)$$

Taking – sign $\Rightarrow$ $x = 1 + \frac{2}{3}$ and $y = 2 - \frac{2}{3}$

$$\Rightarrow R(x, y) = \left(\frac{5}{3}, \frac{4}{3}\right)$$

Q and R are point of contact of tangle of ellipse E_1 and E_2

$$Q = \left(-\frac{a_1^2 m}{c}, \frac{b_1^2}{c}\right) = \left(\frac{1}{3}, \frac{8}{3}\right)$$

$$\Rightarrow \left(-\frac{a_1^2(-1)}{3}, \frac{b_1^2}{c}\right) = \left(\frac{1}{3}, \frac{8}{3}\right)$$

$$\Rightarrow a_1^2 = \; b_1^2 = 8$$

$$a_1^2 = b_1^2(1 - e_1^2)$$

$$\Rightarrow 1 = 8\left(1 - e_1^2\right) \Rightarrow \frac{7}{8}$$

$$\text{For } R = \left(-\frac{a_2^2 m}{c}, \frac{b_2^2}{c}\right) = \left(\frac{5}{3}, \frac{4}{3}\right)$$

$$= \left(-\frac{a_2^2(-1)}{3}, \frac{b_2^2}{3}\right) = \left(\frac{5}{3}, \frac{4}{3}\right)$$

$$\Rightarrow a_2^2 = 5 \text{ and } b_2^2 = 4$$

$$\Rightarrow 4 = 5\,(1 - e_2^2) \Rightarrow e_2^2 = \frac{1}{5}$$

$$e_1^2 + e_2^2 = \frac{7}{8} + \frac{1}{5} = \frac{35+8}{40} = \frac{43}{40}$$

$$e_1 e_2 = \frac{\sqrt{7}}{\sqrt{8}} \times \frac{1}{\sqrt{5}} = \frac{\sqrt{7}}{2\sqrt{10}}$$

55. Eqn. of tangent to hyperbola at (x_1, y_1) is $xx_1 - yy_1 = 1$

$$m = \left(\frac{1}{x_1}, 0\right)$$

Eqn. of normal to hyperbola is

$y_1 x + x_1 y = c$ (passes through (x_1, y_1)

$\Rightarrow$ $x_1y_1 + x_1y_1 = c$

$\Rightarrow$ $y_1 x + xy_1 = 2x_1 y_1$ is also normal to circle

$\Rightarrow$ passes through $(x_2, 0)$

$\Rightarrow$ $y_1 x_2 = 2x_1 y_1$

$\Rightarrow$ $x_2 = 2x_1$

So $M\left(\frac{1}{x_1}, 0\right)$, $N(2x_1, 0)$, $P(x_1, y_1)$

$l = \dfrac{\frac{1}{x_1} + 2x_1 + x_1}{3}$ and $m = y_1$

$l = x_1 + \dfrac{1}{3x_1}$ and $m = \sqrt{x_1^2 - 1}$

$\dfrac{dl}{dx_1} = l - \dfrac{1}{3x_1^2}$ $\qquad$ $\dfrac{dm}{dx_1} = \dfrac{1 \; x_2 x_1}{2\sqrt{x_1^2 - 1}}$

$= \dfrac{x_1}{2\sqrt{x_1^2 - 1}}$

A and B are correct.

56. Let $I = \int_0^{4\pi} e^t \left(\sin^6 a\,t + \cos^4 a\,t\right) dt$

If a is integer then

$\sin^6 a(\pi - t) = \sin a\,t$ and $\cos^4 a(\pi - t) = \cos^4 a\,t$

$I = \int_0^{\pi} e^t \left(\sin^6 a\,t + \cos^4 a\,t\right) dt + \int_{\pi}^{2\pi} e^t \left(\sin^6 a\,t + \cos^4 a\,t\right) dt$

$+ \int_{2\pi}^{3\pi} e^t (\quad) dt + \int_{3\pi}^{4\pi} e^t (\quad) dt$

$I = I_1 + I_2 + I_3 + I_4$

In I_2 sub $x = t - \pi$

$dx = dt$

$I_2 = \int_0^{\pi} e^{\pi + x} \left(\sin^6 ax + \cos^4 a\,x\right) dx$

$= e^{\pi} \int_0^{\pi} e^x \left(\sin^6 ax + \cos^4 a\,x\right) dx$

$= e^{\pi} I_1$

Similarly in I_2 sub. $x = t - 2\pi$ and $= e^{2\pi} I_1$ and I

57. $f'(x) = xF'(x) + F(x)$ ...(1)

Since $F'(x) < 0 \;\; \forall \;\; x \in \left(\frac{1}{2}, 3\right) \Rightarrow F(x)$ is decreasing function

$F(1) = 0$

$\Rightarrow F(x) < 0, \;\; x \in (1, 3)$

Hence from (1)

$f'(x) < 0, \;\; \forall \;\; x \in (1, 3)$

$\Rightarrow$ (A) is correct

f (x) is decreasing function of x

f (1) = 1 × 0 = 0 and f(3) = 3(–4) = –12

so $f(2) < 0 \Rightarrow$ (B) is correct

Since $f'(x) < 0 \;\; \forall \;\; x \in (1, 3)$

$\Rightarrow f'(x) \neq 0$ for any $x \in (1, 3) \Rightarrow$ (C) is correct

From above (D) is correct.

So (A, B, C)

58. $\int_1^3 x^2 F'(x)\, dx = -12$

$\Rightarrow \;\; x^2 F(x)\Big|_1^3 - \int_1^3 2x\, F(x)\, dx = -12$

$$\Rightarrow\ 9(-4)-0-\int_1^3 2x\,F(x)\,dx=-12$$

$$\Rightarrow\ \int_1^3 x\,F(x)\,dx=-12 \qquad \ldots(1)$$

$$\int_1^3 x^3\,F''(x)\,dx=40$$

$$x^3\,F'(x)\Big|_1^3-\int_1^3 3x^2\,F'(x)\,dx=40$$

$$27\,F'(3)-F'(1)-\left[3x^2\,F(x)\Big|_1^3-\int_1^3 6x\,F(x)\,dx\right]=40$$

$$27\,F'(3)-F'(1)-\left[27\,F(3)-3F(1)-6\int_1^3 x\,F(x)\,dx\right]=40$$

$$27\,F'(3)-F'(1)-\left[-108-0-6\int_1^3 x\,F(x)\,dx\right]=40$$

$$27\,F'(3)-F'(1)-[-108-6(-12)]=40 \ \ (\text{from } \ldots(1))$$

$$27\,F'(3)-F'(1)+36=40$$

$$27\,F'(3)-F'(1)=4 \qquad \ldots(2)$$

$$f(x)=x\,F(x)$$

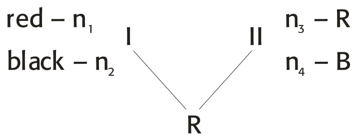
$$f'(x)=xF'(x)+F(x)$$

$$f'(3)=3F'(3)-4 \qquad \ldots(3)$$

$$F'(1)=F'(1\_+0 \qquad \ldots(4)$$

Sub. F′(3) and F′(1) from (3) and (4) into 2
9 [f′(3) + 4] – f′(1) = 4
9 f′(3) – f′(1) + 32 = 0

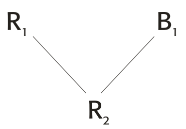
59.



$$P\left(\frac{II}{R}\right)=\frac{P(II)\ P\left(\frac{R}{II}\right)}{P(II)\,P\left(\frac{R}{II}\right)\ +\ P(I)\,P\left(\frac{R}{I}\right)}=\frac{\frac{1}{2}\times\frac{n_3}{n_3+n_4}}{\frac{1}{2}\times\frac{n_2}{n_1+n_2}+\frac{1}{2}\times\frac{n_3}{n_3+n_4}}=\frac{1}{3}\Rightarrow\frac{\frac{n_3}{n_3+n_4}}{\frac{n_2}{n_1+n_2}+\frac{n_3}{n_3+n_4}}=\frac{1}{3}$$

(A and B satisfies)

60.



$$P=P(R_1)\ P\left(\frac{R_2}{R_1}\right)+P(B_1)\ P\left(\frac{R_2}{B_1}\right)$$

$$=\left(\frac{n_1}{n_1+n_2}\right).\left(\frac{n_1-1}{n_1+n_2-1}\right)\ +\ \left(\frac{n_2}{n_1+n_2}\right)\ \frac{n_1}{(n_1+n_2-1)}=\frac{1}{3}$$

$$\Rightarrow\ \frac{n_1^2-n_1\ +\ n_2n_1}{(n_1+n_2)\,(n_1+n_2-1)}=\frac{1}{3}$$

Check from alternatives.
C and D satisfies.