Parabola

Q1 let tangent be $ty = x + at^2$ at $(at^2, 2at)$

$ty = x + 2t^2$

tangent passes through $(2,1)$

$\Rightarrow t(1) = 2 + 2t^2 \Rightarrow 2t^2 - t + 2 = 0$

$D = b^2 - 4ac = 1 - 4 \times 2 \times 2 < 0$

so no real values of $t \Rightarrow$ no tangent.

Q2 Point $(4,-4)$ lies on the curve $y^2 = 4x$

let $(4,-4) = (at^2, 2at) = (t^2, 2t) \Rightarrow t = -2$

Eqn. of normal at $(at^2, 2at)$ is

$y = -tx + 2at + at^3$

$y = -(-2)x + 2(1)(-2) + 1(-2)^3$

$y = 2x - 12$

$\Rightarrow k = -12$

Q3 let tangent be $ty = (x-a) + at^2$

passes through $(0,0)$

$\Rightarrow 0 = 0 - a + at^2 \Rightarrow t = \pm 1$

Eqn. of tangent $\pm y = x - a + a(\pm 1)^2$

$\Rightarrow \pm y = x$

These lines are $\perp \Rightarrow$ angle is $90°$

Q4 $y^2 - 4y = 6x - 13$

$y^2 - 4y + 4 = 6x - 13 + 4$

$(y-2)^2 = 6x - 9 = 6(x - 3/2)$, $4a = 6 \Rightarrow a = 3/2$

$(x - 3/2, y - 2)_f = (3/2, 0)$

$(x, y)_f = (3, 2)$

Q5 $y^2 = kx - 8 = k(x - 8/k)$, $4a = k$

Eqn. of Directrix $x - 8/k + \frac{k}{4} = 0$

$x = \frac{8}{k} - \frac{k}{4}$

Directrix is $x = 1 \Rightarrow \frac{8}{k} - \frac{k}{4} = 1$

$32 - k^2 = 4k$

$\Rightarrow k^2 + 4k - 32 = 0$

$k = -8, 4$

Q6 let tangent to parabola $y^2 = 4x$ be

$ty = x + t^2$ is also

tangent to circle $\Rightarrow p = r$

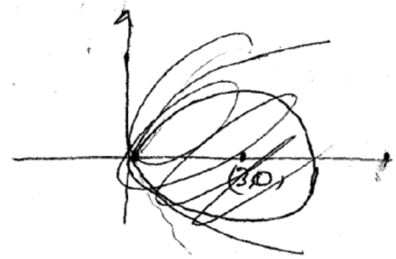
center $(3,0)$, $r = 3$

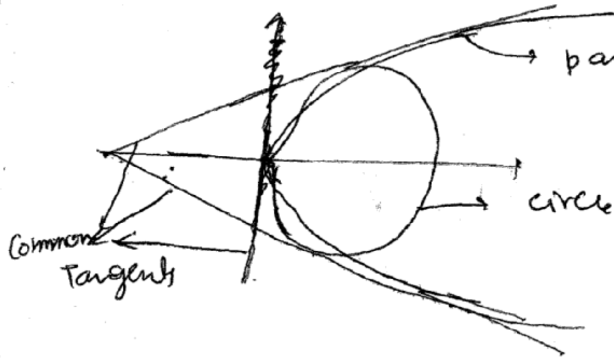
line $x - ty + t^2 = 0$

$$p = \frac{|3 - 0 + t^2|}{\sqrt{1+t^2}} = 3 \ (= r)$$

$$\Rightarrow (t^2+3)^2 = 9(1+t^2) \Rightarrow t^4 + 9 + 6t^2 = 9 + 9t^2$$

$$\Rightarrow t^4 = 3t^2 \Rightarrow t = 0, \pm\sqrt{3}$$





common tangent above the x have the slope (from the graph)

so required tagent is

$$\sqrt{3}\,y = x + 3$$

Q7 If normal makes equal intercept from axis $\Rightarrow$ slope $= \pm 1$

Eqn. of normal is $y = -tx + 2at + at^3$

slope $= -t = \pm 1$

$\Rightarrow t = \pm 1$

points on curve are $(at^2, 2at)$

$(1,2)$ & $(1,-2)$

Q8 Center of circle $\left(\frac{p}{2}, 0\right)$

Directrix of parabola $x + p/2 = 0$ = Tangent to circle.

perpendicular distance from $(p/2, 0)$ to tangent $= \frac{|p/2 + p/2|}{1} = p$

$\Rightarrow$ rad. of circle $= p$

Eqn. of circle $(x - p/2)^2 + y^2 = p^2$ — (1)

For intersection of circle with parabola $y^2 = 2px$

sub y^2 in (1) $\Rightarrow (x - p/2)^2 + 2px = p^2$

$\Rightarrow (x + p/2)^2 = p^2$

$\Rightarrow x + p/2 = \pm p$

2

$\Rightarrow x = p/2$ or $-\frac{3p}{2}$ (not possible)

Sub. x in $y^2 = 2px$

$y^2 = 2p \cdot p/2 \Rightarrow y = \pm p$

$\Rightarrow$ point of intersection $(p/2, \pm p)$

Q9 $y = -tx + 2at + at^3$

$-t = 1, \ a = 2$

$y = x - 4 - 2 \Rightarrow y = x - 6$

Q10 Eqn of tangent $ty = x + at^2$

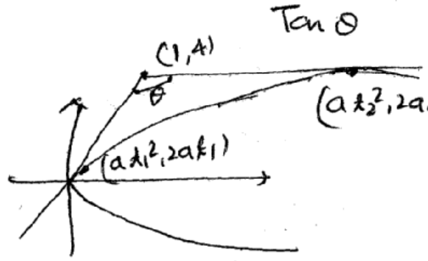
passes through $(1, 4)$

$\Rightarrow 4t = 1 + t^2 \Rightarrow t^2 - 4t + 1 = 0$

suppose two values of t are t_1 & t_2 then slope of tangents $(1/t)$ are t_1 & t_2

Angle between tangents is θ then

$$\tan\theta = \left|\frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1}\times\frac{1}{t_2}}\right| = \left|\frac{t_2 - t_1}{t_1 t_2 + 1}\right|$$

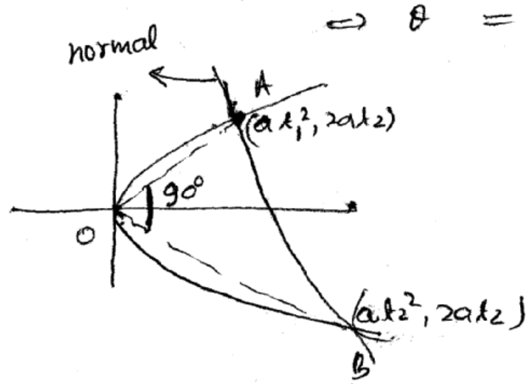


$$= \frac{\sqrt{(-4)^2 - 4\times1\times1}}{1+1}$$

$$= \frac{\sqrt{12}}{2} = \sqrt{3}$$

$$\Rightarrow \theta = \pi/3$$

Q11



$t_2 = -t_1 - \frac{2}{t_1}$

$m_{OA} = \frac{2at_1 - 0}{at_1^2 - 0} = \frac{2}{t_1}$

$m_{OB} = \frac{2}{t_2}$

$m_{OA}\, m_{OB} = -1$

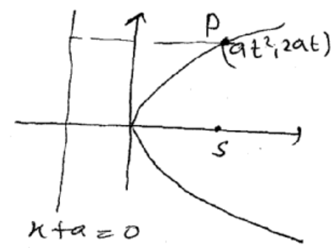
$\Rightarrow \left(\frac{2}{t_1}\right)\left(\frac{2}{t_2}\right) = -1 \Rightarrow t_1 t_2 = -4$

$t_1\left(-t_1 - \frac{2}{t_1}\right) = -4 \Rightarrow t_1 = \pm\sqrt{2}$

Slope of normal $= -t_1 = \mp\sqrt{2}$

Q12 let point be $P(at^2, 2at)$

distance of point from focus = distance from the directrix



$$= \frac{|a+at^2|}{\sqrt{1^2+0}} = a+at^2$$

If $a+at^2$ is minimum w.r.t. to t then $t = 0$

so ~~distance~~ minimum distance is a & point is $(0,0)$

Q13 $y^2 = 4a(x-b)$

Eqn of tangent line is ~~to~~ of the form

$ty = x-b + at^2$ — ①

while tangent line is $lx + my + n = 0$

$\Rightarrow my = -lx - n$ — ②

① & ② are same line

$$\Rightarrow \frac{t}{m} = \frac{1}{-l} = \frac{-b+at^2}{-n}$$

$$\Rightarrow t = -\frac{m}{l}, \quad \& \quad \frac{n}{l} = -b + at^2$$

$$\Rightarrow \frac{n}{l} = -b + a\left(-\frac{m}{l}\right)^2$$

$$\Rightarrow nl = -bl^2 + am^2$$

Alt. Solve line $lx + my + n = 0$ with $y^2 = 4a(x-b)$ — ①

$lx + my + n = 0$

$\Rightarrow x = -\frac{1}{l}(my+n)$ & sub in ①

$$\Rightarrow y^2 = 4a\left[-\frac{1}{l}(my+n) - b\right]$$

$$y^2 = -\frac{4am}{l}y - \frac{4an}{l} + 4ab$$

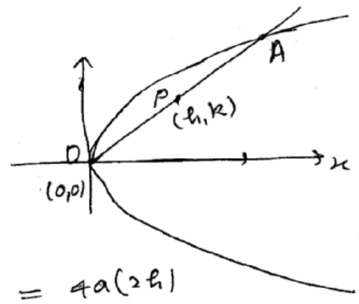
$$y^2 + \frac{4am}{l}y + \frac{4an}{l} - 4ab = 0$$

If touches then only one point of intersection $\Rightarrow D = 0$

$$\frac{16a^2m^2}{l^2} - 4\times 1\left[\frac{4an}{l} - 4ab\right] = 0$$

14 Directrix of parabola.

15 let OA is one of chord whose middle point is (h, k)



O is $(0,0)$ & P is (h, k)

$\Rightarrow A = (2h, 2k)$

A lies on the parabola $\Rightarrow (2k)^2 = 4a(2h)$

$$\Rightarrow k^2 = 2ah$$
$$y^2 = 2ax$$

16 Eqn. of normal $y = -tx + 2\left(\frac{1}{4}\right)t + \frac{1}{4}t^3$

passes through $(c, 0)$

$$\Rightarrow 0 = -tc + \frac{t}{2} + \frac{1}{4}t^3$$
$$0 = t\left[-c + \frac{1}{2} + \frac{1}{4}t^2\right]$$
$$\Rightarrow t = 0 \quad \text{or} \quad -c + \frac{1}{2} + \frac{1}{4}t^2 = 0$$
$$\Rightarrow \frac{1}{4}t^2 = c - \frac{1}{2}$$
$$t = \pm 2\sqrt{c - \tfrac{1}{2}}$$

Two of the normals are $\perp$ $\Rightarrow m_1 m_2 = -1$

Slope of normal is $(-t)$ $\Rightarrow (-t_1)(-t_2) = -1$

$$\Rightarrow \left(-2\sqrt{c - \tfrac{1}{2}}\right)\left(2\sqrt{c - \tfrac{1}{2}}\right) = -1$$
$$\Rightarrow c - \frac{1}{2} = \frac{1}{4} \Rightarrow c = \frac{3}{4}$$

17 let tangent to $y^2 = 8x$ be $ty = x + 2t^2$ is also tangent to $xy = -1$

Solving tangent & curve

$y = -\frac{1}{x}$ sub. in Eqn. of tangent

$$-\frac{t}{x} = x + 2t^2 \Rightarrow -t = tx^2 + 2t^2 x$$
$$\Rightarrow x^2 + 2t^2 x + t = 0$$
$$D = 0 \Rightarrow 4t^4 - 4t = 0$$
$$\Rightarrow t = 0, 1$$

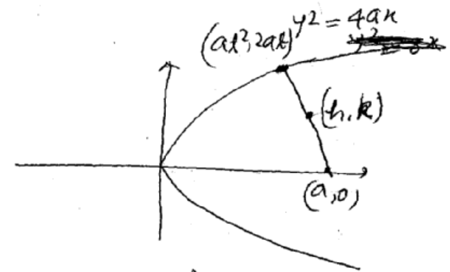
So common tangent is $x = 0$ or $y = x + 2$

18

$$h = \frac{a + at^2}{2} \quad — (1)$$

$$k = \frac{2at + 0}{2} = at \quad — (2)$$

$$\Rightarrow t = k/a \quad \text{sub. in } (1)$$



$$\Rightarrow 2h = a + a\,k^2/a^2 \Rightarrow 2h = a + k^2/a$$

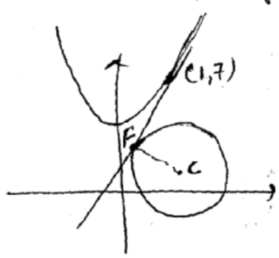
$$\Rightarrow k^2 = a(2h - a)$$

$$y^2 = a(2x - a) = 2a(x - a/2)$$

So Locus is $y^2 = 2a(x - \frac{a}{2})$

Eqn of directrix. $x - a/2 + a/2 = 0$

19



Eqn of tangent

$$\tfrac{1}{2}(y+7) = 1\cdot x + 6$$

$$y + 7 = 2x + 12$$

$$\Rightarrow y = 2x + 5$$

Point of contact of this line from circle is foot of perpendicular from center $(-8,-6)$ to tangent.

let foot be $(\beta, 2\beta + 5)$

$$m_{FC} \cdot m_T = -1$$

$$\cancel{\left(\frac{2\beta+5+8}{\beta+6}\right) \times 2 = -1} \Rightarrow \cancel{4\beta + 22 = -\beta + 11}$$

$$\cancel{-5\beta}$$

$$\left(\frac{2\beta+5+6}{\beta+8}\right) \times 2 = -1$$

$$\Rightarrow 4\beta + 22 = -\beta - 8 \Rightarrow 5\beta = -30$$

$$\beta = -6$$

so foot is $(-6, -12+5) = (-6,-7)$

20 Intersection of two curves

$$x^2 + 4x - 6 + 1 = 0$$

$$x^2 + 4x - 5 = 0$$

$$x = -5, 1$$

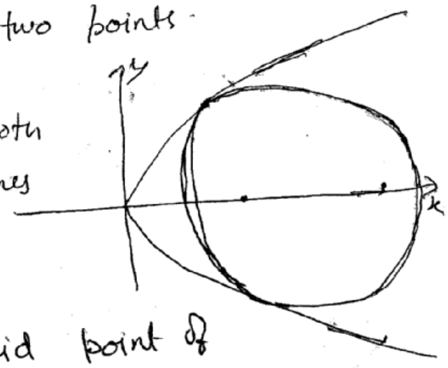
$x = -5$ is not possible (parabola is not defined for $-$ve x)

so $x = 1 \Rightarrow y = \pm 2$

both circle & parabola are symmetric about x axis (4)

$\Rightarrow$ they touch ~~each~~ each other at two points.

~~Alt~~ ~~Eqn~~ Eqn of tangent at (1,2) to both curve is same $\Rightarrow$ curve touches each other.



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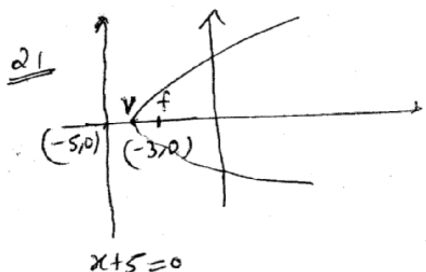


Vertex is mid point of focus & intersection of directix & axis.

$V = \left(\frac{-5+3}{2}, 0\right) = (-4,0)$

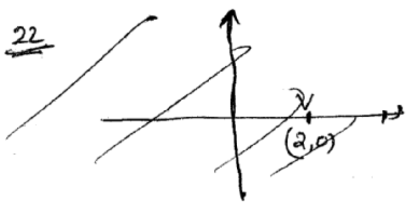
Eqn of parabola $y^2 = 4a(x-(-4))$

$y^2 = 4a(x+4)$

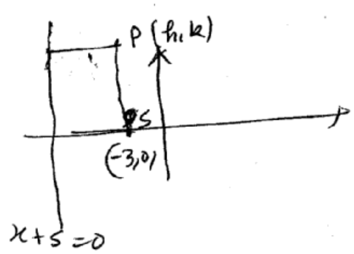
a is distance bt. Vertex & focus $= 1$

Eqn. $y^2 = 4(x+4)$

22



Alt.



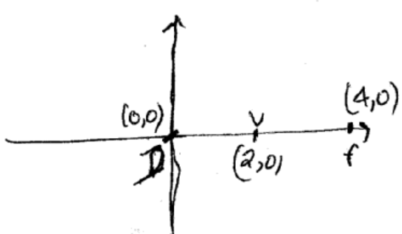
P lies such that PS = PM

$\sqrt{(h+3)^2+k^2} = |h+5| \Rightarrow (x+3)^2 + y^2 = (x+5)^2$

$y^2 = 4x+16$

$y^2 = 4(x+4)$

22



Vertex (2,0)

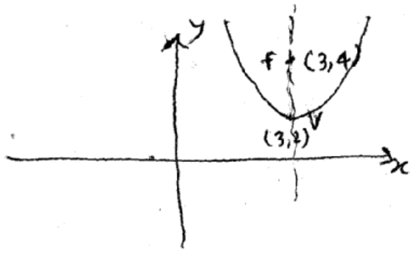
Directrix $x=0$

$\Rightarrow$ axis is x axis. Intersection of directrix & axis $= (0,0)$

Vertex is mid point of focus & D $\Rightarrow$ focus (4,0)

23 $y^2 - 12x - 4y + 4 = 0 \Rightarrow (y-2)^2 = 12x$

$(x, y-2)_f = (3,0) \Rightarrow (x,y)_f = (3,2)$



Eqn of parabola

$(x-3)^2 = 4a(\cancel{x}\, y-2)$

$(x-3)^2 = 4(2)(y-2)$

$x^2 + 9 - 6x = 8y - 16$

$x^2 - 6x - 8y + 25 = 0$

a = distance bt. vertex & focus

$a = 2$

24 (a) $\frac{x}{3} = \cos t$, $\frac{y}{4} = \sin t$, $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ not parabola

(b) $\frac{x^2-2}{2} = \cos t$, $y = 4\cos^2(t/2) = 2(1+\cos t) \Rightarrow$ $y = 2\left[1 + \frac{x^2-2}{2}\right]$

$\Rightarrow$ parabola.

(c) $x = \tan^2 t$, $y = \sec^2 t$, $\sec^2 t - \tan^2 t = 1 = y - x$

(d) $x^2 = 1 - \sin t$, $y^2 = \left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)^2 = 1 + \sin t$

$\Rightarrow x^2 + y^2 = 2 \Rightarrow$ (circle.)

25 let Point $(3t^2, 6t)$, focus $(3,0)$

$d = \sqrt{(3t^2-3)^2 + (6t)^2} = 4$

$\Rightarrow 9\left[(t^2-1)^2 + 4t^2\right] = 16$

$\Rightarrow (t^2+1)^2 = \frac{16}{9} \Rightarrow t^2 + 1 = \pm\frac{4}{3}$, ~~$-$ is not possible.~~

~~$\Rightarrow$ not $\Rightarrow$ no possible t.~~

~~26~~

$t^2 + 1 = 4/3$

$t^2 = 1/3$, $t = \pm\frac{1}{\sqrt{3}}$

point $\left(3 \times \frac{1}{3}, 6 \times \left(\pm\frac{1}{\sqrt{3}}\right)\right)$

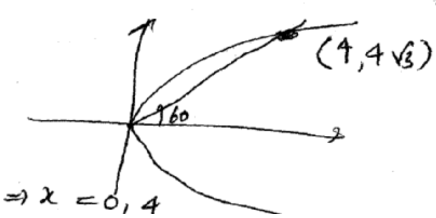
$= (1, \pm 2\sqrt{3})$

26 Eqn of chord: $y = (\tan 60^\circ)x$

$y = \sqrt{3}x$

solving with parabola $y^2 = 12x$

$3x^2 = 12x \Rightarrow x = 0, 4$



(5)

$x = 0 \Rightarrow y = 0, \quad x = 4 \Rightarrow y = 4\sqrt{3}$

length of chord $= \sqrt{(4-0)^2 + (4\sqrt{3}-0)^2} = 8$

27 slope of chord $= \frac{1-0}{4-3} = 1 =$ slope of tangent.

Eqn of tangent to $(x-3)^2 = 4 \cdot \frac{1}{4} y$

$$t(x-3) = y + \frac{1}{4}t^2$$

slope of tangent is $1 \Rightarrow t = 1$

Eqn. of tangent $1(x-3) = y + \frac{1}{4}$

$$y = x - \frac{13}{4}$$

28 Eqn. of normal to $y^2 = -8x$

$$y = tx + 2\times 2t + 2t^3$$

$$y = tx + 4t + 2t^3 \quad —(1)$$

Normal is $2x + y + d = 0 \quad —(2)$

$$\Rightarrow y = -2x - d$$

(1) & (2) are same

$$\Rightarrow \frac{1}{1} = \frac{t}{-2} = \frac{4t + 2t^3}{-d}$$

$$\Rightarrow t = -2 \quad \& \quad 1 = \frac{-8-16}{-d}$$

$$d = 24$$

(29)

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2$$

$$y = \frac{a^3}{3}\left[x^2 + \frac{3}{2a}x\right] - 2$$

$$y = \frac{a^3}{3}\left[x^2 + \frac{3}{2a}x + \frac{9}{16a^2}\right] - \frac{a^3}{3}\cdot\frac{9}{16a^2} - 2a$$

$$y = \frac{a^3}{3}\left[x + \frac{3}{4a}\right]^2 - \frac{3}{16}a - 2$$

$$y + \frac{3}{16}a + 2 = \frac{a^3}{3}\left[x + \frac{3}{4a}\right]^2$$

$$\left[x + \frac{3}{4a}\right]^2 = \frac{3}{a^3}\left[y + \frac{3}{16}a + 2a\right]$$

$$\text{Vertex} = \left(-\frac{3}{4a}, \frac{3}{16}a + 2a\right)$$

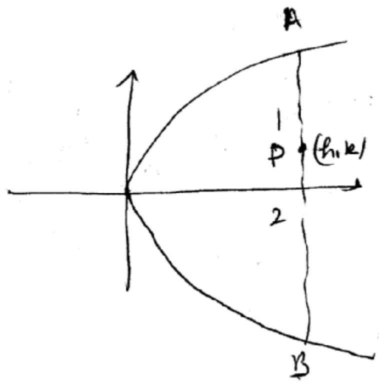
let Vertex (h, k)

$\Rightarrow h = \frac{-3}{4a}$, $k = \frac{35a}{16}$

$\Rightarrow a = -\frac{3}{4h} \Rightarrow k = \frac{35}{16}\left(\frac{-3}{4h}\right)$

$\Rightarrow hk = -\frac{105}{64}$

$\Rightarrow$ Locus is $xy = -\frac{105}{64}$



30 let AB be one of double ordinate ~~(A line ⊥ to axis of parabola)~~ (Intercept of line ⊥ to axis of parabola)

let $A = (at^2, 2at)$, $B = (at^2, -2at)$

If P trisects AB then $h = at^2$ —(1)

$k = \frac{2(2at) + 1(-2at)}{2+1}$

$= \frac{2at}{3}$

$\Rightarrow t = \frac{3k}{2a}$ sub. in (1)

$h = a\left(\frac{9k^2}{4a^2}\right)$

$x = \frac{9}{4}\frac{y^2}{a} \Rightarrow y^2 = \frac{4}{9}ax$

(31) Both lines are ~~drawn~~ drawn from $(-a, -b)$ so these are tangents drawn from $(-a, -b)$

let tangents be $ty = x + at^2$ (In terms of parameter)

$y = \frac{1}{t}x + at$

slope is $\frac{1}{t} = m$

$\Rightarrow$ Tangent line in terms of slope $y = mx + \frac{a}{m}$

passes through $(-a, -b)$

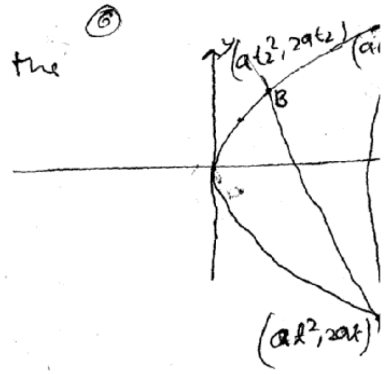
$\Rightarrow -b = -ma + \frac{a}{m}$

$\Rightarrow -bm = -m^2a + a$

$\Rightarrow m^2a - bm - a = 0$

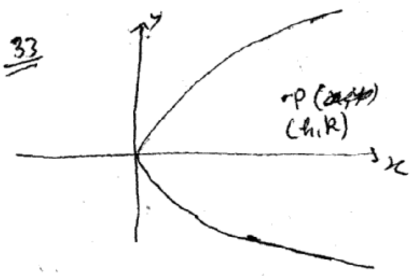
let two values of m are m_1 & m_2 $\Rightarrow m_1 + m_2 = b/a$, $m_1 m_2 = \frac{-a}{a} = -1$

32 let the normal at A & B intersects at the parabola at C, then (6)



$$t = -t_1 - \frac{2}{t_1} \quad —(1)$$
$$\& \quad t = -t_2 - \frac{2}{t_2} \quad —(2)$$
$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$
$$t_2 - t_1 + \frac{2}{t_2} - \frac{2}{t_1} = 0$$
$$t_2 - t_1 + 2\left(\frac{t_1 - t_2}{t_1 t_2}\right) = 0$$
$$\frac{2(t_1 - t_2)}{t_1 t_2} = t_1 - t_2 \Rightarrow \frac{2}{t_1 t_2} = 1$$
$$t_1 t_2 = 2$$

33

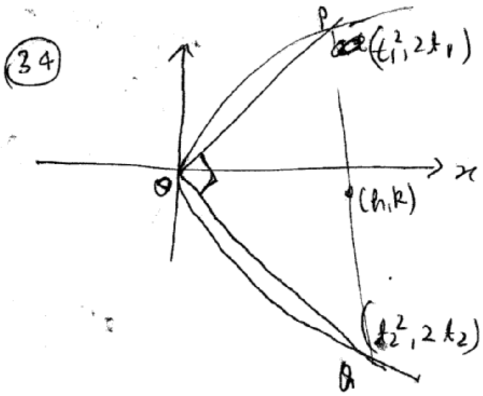


Let Eqn. of normal be
$$y = -tx + 2at + at^3$$
Normal passes through (h, k)
$$k = -th + 2at + at^3$$
$$at^3 + (2a-h)t - k = 0 \quad \text{(suppose three roots are } t_1, t_2, t_3)$$
$$t_1 + t_2 + t_3 = 0 \quad —(1)$$
$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a} \quad —(2)$$
$$t_1t_2t_3 = \frac{k}{a} \quad —(3)$$

If two of the normals coincide $\Rightarrow t_1 = t_2$

Sub. t_2 in (1)
$$2t_1 + t_3 = 0 \Rightarrow t_3 = -2t_2$$
Sub. t_2 & t_3 in (2) & (3)
$$t_1^2 - 2t_1^2 - 2t_1^2 = \frac{2a-h}{a}$$
$$-3t_1^2 = \frac{2a-h}{a} \Rightarrow t_1^2 = \frac{2a-h}{-3a} \quad —(4)$$
$$\& \quad -2t_1^3 = \frac{k}{a} \Rightarrow t_1^3 = \frac{k}{-2a} \quad —(5)$$
cubing (4)th & squaring (5)
$$\Rightarrow t^6 = \frac{(2a-h)^3}{-27a^3} \quad \& \quad t_1^6 = \frac{k^2}{4a^2}$$
$$\Rightarrow \frac{(2a-h)^3}{-27a^3} = \frac{k^2}{4a^2}$$
$$\Rightarrow (2a-h)^3 = -\frac{27ak^2}{4} \Rightarrow 4(x-2a)^3 = 27ay^2$$

(34)



Let P & Q be $(t_1^2, 2t_1)$ & $(t_2^2, 2t_2)$

$m_{OP} \cdot m_{OQ} = -1$

$\left(\frac{2}{t_1}\right)\left(\frac{2}{t_2}\right) = -1$

$\Rightarrow t_1 t_2 = -4$

Let mid point of PQ be (h,k)

$\Rightarrow h = \frac{t_1^2 + t_2^2}{2}$ —①, $k = \frac{2t_1 + 2t_2}{2} = t_1 + t_2$ —②

From ① $t_1^2 + t_2^2 = 2h$

$(t_1+t_2)^2 - 2t_1t_2 = 2h$

$k^2 - 2(-4) = 2h$

$\Rightarrow k^2 + 8 = 2h \Rightarrow y^2 + 8 = 2x$

36

$25\left[(x-2)^2 + (y-3)^2\right] = (3x-4y+7)^2$

$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = \frac{|3x-4y+7|}{5}$

$\Rightarrow$ distance of point (x,y) from $(2,3)$ & line $3x-4y+7=0$ are equal $\Rightarrow$ Locus of (x,y) is parabola with focus $(2,3)$ & line $3x-4y+7=0$ directrix

Distance of focus from directrix is $= 2a = \frac{|3\times2 - 4\times3 + 7|}{5}$

$= \frac{1}{5}$

length of latus rectum $= 4a = \frac{2}{5}$

37 Solving the line with curve

$\Rightarrow (2x-3)^2 = 4a(x - \frac{1}{3})$

$\Rightarrow 4x^2 + 9 - 12x = 4ax - \frac{4a}{3}$

$\Rightarrow 4x^2 - (12+4a)x + 9 + \frac{4a}{3} = 0$

$D = 0 \Rightarrow (12+4a)^2 - 4\cdot 4\left(9 + \frac{4a}{3}\right) = 0$

$\Rightarrow (2a+6)^2 - 9 - \frac{4a}{3} = 0$

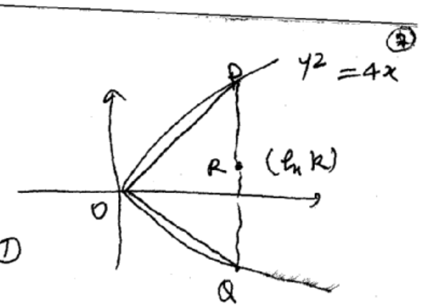
$\Rightarrow 4a^2 + 36 + 24a - \frac{4a}{3} - 9 = 0$

$\Rightarrow 4a^2 + \frac{68a}{3} + 27 = 0$

34 let P & Q be $(t_1^2, 2t_1)$ & $(t_2^2, 2t_2)$

$m_{OP} \cdot m_{OQ} = -1$

$\left(\frac{2t_1 - 0}{t_1^2 - 0}\right)\left(\frac{2t_2 - 0}{t_2^2 - 0}\right) = -1 \Rightarrow t_1 t_2 = -4$ —(1)



(h, k) is middle point of P, Q

$\Rightarrow h = \frac{t_1^2 + t_2^2}{2}$ —(2) & $k = \frac{2t_1 + 2t_2}{2}$ —(3)

from (3) $t_1 + t_2 = k$

on squaring both side

$\Rightarrow t_1^2 + t_2^2 + 2t_1 t_2 = k^2$

$\Rightarrow 2h + 2(-4) = k^2$

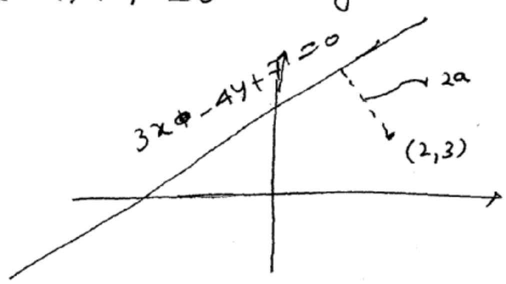
$\Rightarrow 2x - 8 = y^2$

35 $25\left[(x-2)^2 + (y-3)^2\right] = (3x - 4y + 7)^2$

$\Rightarrow (x-2)^2 + (y-3)^2 = \left(\frac{3x - 4y + 7}{5}\right)^2$

$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = \frac{|3x - 4y + 7|}{5}$

$\Rightarrow$ distance of (x, y) from (2, 3) is equal to distance from line $3x - 4y + 7 = 0$ $\Rightarrow$ (x, y) lies on parabola with (2, 3) focus & $3x - 4y + 7 = 0$ being directrix of parabola.



In parabola ⊥ distance from focus to directrix is half the latus rectum.

$2a = \frac{|3 \times 2 - 4 \times 3 + 7|}{\sqrt{3^2 + 4^2}} = \frac{1}{5}$

length of latus rectum $= 4a = \frac{2}{5}$

(37) Solving the tangent with curve.

$(2x-3)^2 = 4a(x-1/3)$

$\Rightarrow 4x^2 - 12x + 9 = 4ax - \frac{4a}{3} \Rightarrow 4x^2 - (12+4a)x + 9 + \frac{4a}{3} = 0$

Since line is tangent $\Rightarrow D = 0$

$\Rightarrow (12+4a)^2 - 4\times 4\left(9 + \frac{4a}{3}\right) = 0$

$\Rightarrow 16\left[(a+3)^2 - 9 - \frac{4a}{3}\right] = 0$

$\Rightarrow \left[a^2 + 9 + 6a - 9 - \frac{4a}{3}\right] = 0$

$\Rightarrow a^2 + \frac{14a}{3} = 0 \Rightarrow a = 0, \frac{-14}{3}$

but $a \neq 0 \Rightarrow a = -14/3$

Alternative let tangent to $y^2 = 4a(x-1/3)$ be

$ty = x - 1/3 + at^2$ — (1)

tangent line is $y = 2x-3$ — (2)

(1) & (2) are same

$\Rightarrow \frac{t}{1} = \frac{1}{2} = \frac{at^2 - 1/3}{-3}$

$\Rightarrow t = 1/2, \quad -3/2 = at^2 - 1/3$

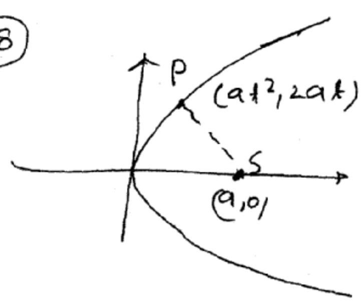
~~$-2/3 + 1/3$ a~~

$-3/2 + \frac{1}{3} = a\left(\frac{1}{2}\right)^2$

$\Rightarrow \frac{-9+2}{6} = \frac{a}{4}$

$\Rightarrow a = \frac{-14}{3}$

(38)



let parabola be $y^2 = 4ax$

& P point be $(at^2, 2at)$

Eqn. of circle with PS as diameter.

$(x-a)(x-at^2) + (y-0)(y-2at) = 0$

$\Rightarrow x^2 + y^2 - (a+at^2)x - 2aty + a^2t^2 = 0$

Center of circle $= \left(\frac{at^2+a}{2}, \frac{2at+0}{2}\right) = \left(\frac{a(1+t^2)}{2}, at\right)$

radius $= \frac{1}{2}\sqrt{(at^2-a)^2+(2at)^2} = \frac{1}{2}(at^2+a)$

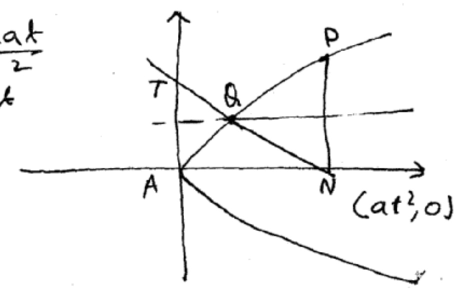
Since rad. $=$ x co-ordinate of center of circle

$\Rightarrow$ circle touches the y axis $\Rightarrow$ ~~Tangent~~ tangent to parabola at the vertex is also tangent to the required circle.

(39) let $P(at^2, 2at)$

Eqn of $\perp$ bisector of PN $y = \frac{2at}{2}$

$y = at$



Intersection of this line with parabola is Q.

$y^2 = 4ax$

$a^2t^2 = 4ax \Rightarrow x = \frac{1}{4}at^2$

so Q $\left(\frac{1}{4}at^2, at\right)$

Eqn of QN $\frac{y-0}{x-at^2} = \frac{at-0}{\frac{1}{4}at^2-at^2} = -\frac{4}{3t}$

$\Rightarrow y = -\frac{4}{3t}(x-at^2)$

For T point $x = 0 \Rightarrow y = \frac{4}{3}at$

$\Rightarrow T = \left(0, \frac{4}{3}at\right)$

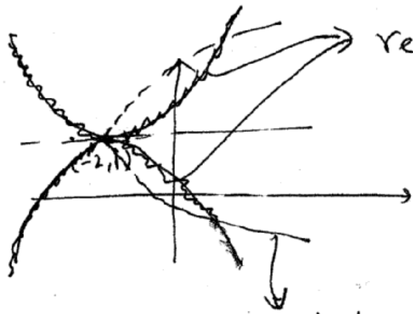
$AT = \frac{4}{3}at$, $NP = 2at$

$\Rightarrow$ ~~$AT = \frac{3}{4}NP$~~ $\Rightarrow \frac{AT}{NP} = k = \frac{\frac{4}{3}at}{2at}$

$= 2/3$

40 $(y-1)^2 = 4(x+2)$, $A = (-2, 1)$

$L = 4$

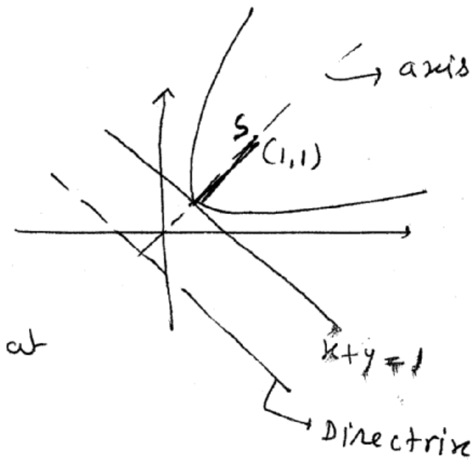


→ required parabola.

Eqn. $(x+2)^2 = \pm 2\times 4(y-1)$

$x^2+4+4x = \pm 8(y-1)$

↓ Original parabola



(41) ~~Axis~~ Directrix is ~~∥ to~~ parallel to tangent at it's vertex.

Let Directrix be $x+y+c=0$

Distance of focus from tangent at vertex $= \dfrac{|1+1-1|}{\sqrt{1+1}} = \dfrac{1}{\sqrt{2}}$

Distance between directrix & tangent will also be $\dfrac{1}{\sqrt{2}}$

$\Rightarrow \dfrac{|c-1|}{\sqrt{1+1}} = \dfrac{1}{\sqrt{2}} \Rightarrow |c-1| = 1$

$\Rightarrow c-1 = \pm 1 \Rightarrow c = 2, 0$

If c is 2 then directrix passes through (1,1) which is not possible

so $c=0$

Eqn. of directrix $x+y=0$

Eqn of parabola using $PS = PM$

$$\sqrt{(h-1)^2+(k-1)^2} = \frac{|h+k|}{\sqrt{2}}$$

$$\Rightarrow (x-1)^2+(y-1)^2 = \frac{(x+y)^2}{2} \quad (1)$$

~~length~~ value of $a = \dfrac{1}{\sqrt{2}}$, length of latus rectum $= 4a$
$= 4\cdot\dfrac{1}{\sqrt{2}}$
$= 2\sqrt{2}$

Vertex is intersection of axis & tangent at the vertex.

Eqn of axis $x-y+c=0$ ($\perp$ to tangent & passes through (1,1))

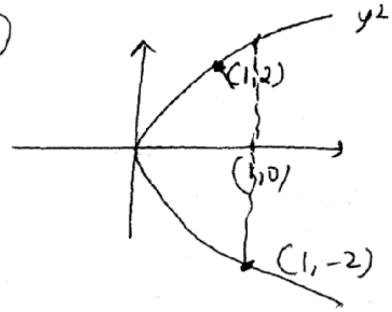
$\Rightarrow$ ~~x-y =~~ $1-1+c=0 \Rightarrow c=0$

Eqn of axis $x-y=0$

their intersection is $(\frac{1}{2},\frac{1}{2})$

Simplify Eq(1) & alternative (b) both are same.

(42)



~~Eqn~~ chord passes through (1,2) & (1,0) $\Rightarrow$ Eqn of chord is $x=1$

$\Rightarrow$ other ends of chord is (1,−2)

(1,−2) satisfies the alternative (a), (b), (d)

(43) let P be (α,α) lies on parabola $\Rightarrow \alpha^2 = 4\alpha \Rightarrow \alpha = 4$

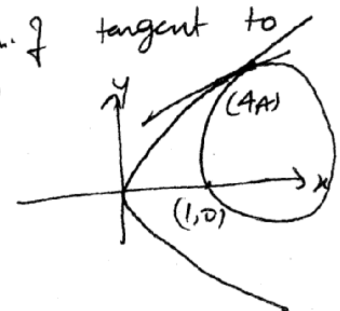
so $P(4,4)$

Circle touches the parabola at (4,4) Eqn of tangent to parabola at (4,4) is

$4y = \frac{1}{2}\cdot 4(x+4)$

$2y = x+4$

$\Rightarrow$ Tangent to parabola is also tangent to circle.



Eqn of circle can be taken as.

$(x-4)^2 + (y-4)^2 + \lambda(x-2y+4) = 0$

~~Eq~~ Above circle passes through (1,0) $\Rightarrow$

$(1-4)^2 + (0-4)^2 + \lambda(1+4) = 0$

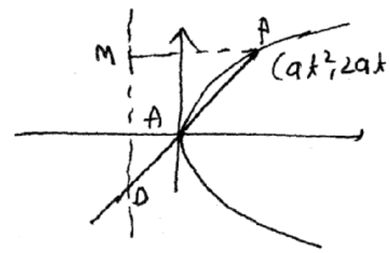
$\Rightarrow \lambda = -5$

so Eqn of circle is $(x-4)^2 + (y-4)^2 - 5(x-2y+4) = 0$

$x^2+y^2-13x+2y+12=0$

(44) Similar to Q. 18.

(45) let P be $(at^2, 2at)$



M $(-a, 2at)$

Eqn. of A.P. $\frac{y-0}{x-0} = \frac{2at}{at^2}$

$\Rightarrow y = \frac{2}{t} x$

D is intersection of Directrix $x = -a$ & above line

$\Rightarrow D = \left(-a, -\frac{2a}{t}\right)$

Eqn. of circle with MD as diameter

$(x+a)(x+a) + (y-2at)\left(y + \frac{2a}{t}\right) = 0$

Intersection of circle with x axis $\Rightarrow y = 0$

$\Rightarrow (x+a)(x+a) - 2at \times \frac{2a}{t} = 0$

$\Rightarrow (x+a)^2 = 4a^2 \Rightarrow x + a = \pm 2a$

$\Rightarrow x = a, -3a$

(46) let Eqn. to tangent to $x^2 = y$ be

$t_1 x = y + \frac{1}{4} t_1^2$ — (1)

Eqn of tangent to curve $y = -(x-2)^2$ be

$t_2(x-2) = -y + \frac{1}{4} t_2^2$ — (2)

(1) & (2) are same lines

$\Rightarrow \frac{t_1}{t_2} = \frac{1}{-1} = \frac{\frac{1}{4} t_1^2}{\frac{1}{4} t_2^2 + 2t_2}$

$\Rightarrow t_1 = -t_2$ & $-\frac{1}{4} t_1^2 = \frac{1}{4} t_2^2 + 2t_2$

$\Rightarrow -\frac{1}{4} t_1^2 = \frac{1}{4}(-t_1)^2 + 2(-t_1)$

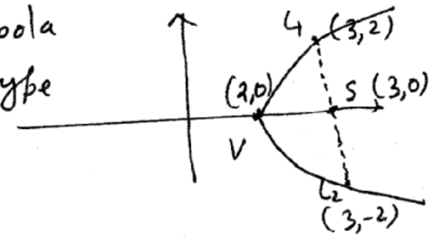
$\Rightarrow -\frac{1}{2} t_1^2 = -2t_1$

$\Rightarrow t_1 = 0, 4$

so common tangent is $0 = y$ &

$4x = y + 4$

(47) From figure required parabola is standard parabola of first type with origin shifted at (2,0)

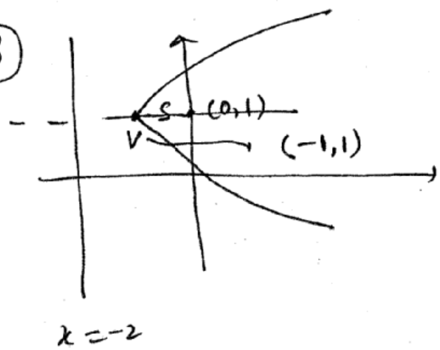


$(y-0)^2 = 4a(x-2)$

$4a = 4$ (length of LR)

$y^2 = 4(x-2)$

(48)



Eqn of parabola

$(y-1)^2 = 4a(x+1)$

$a = 1$ (distance between Vertex & focus)

$(y-1)^2 = 4(x+1)$

Points $x+1 = 1 \times t^2$

$y-1 = 2 \times 1\, t$

(49) let Eqn of tangent $ty = x + at^2$

$y = \frac{1}{t}x + at$

$\frac{1}{t} = m$ so in terms of slope $y = mx + \frac{a}{m}$

Tangent passes through (α,β) $\Rightarrow$ $\beta = m\alpha + \frac{a}{m}$

$\Rightarrow \beta m = m^2\alpha + a$

$m^2\alpha - m\beta + a = 0$

slope of one is double the other

let slope be $m_1, 2m_1$

$m_1 + 2m_1 = \beta/\alpha$, $m_1 \cdot 2m_1 = \frac{a}{\alpha}$

$\Rightarrow m_1 = \frac{\beta}{3\alpha}$ $\Rightarrow$ $2\left(\frac{\beta}{3\alpha}\right)^2 = \frac{a}{\alpha}$

$\Rightarrow \frac{2\beta^2}{9\alpha^2} = \frac{a}{\alpha}$

$\Rightarrow 2\beta^2 = 9a\alpha$

here $a = 1$ $\Rightarrow 2\beta^2 = 9\alpha$

(50) Pt. of intersection

$x^2 + \frac{3}{2}(x+1) + 2x = 0 \Rightarrow 2x^2 + 7x + 3 = 0$

$2x^2 + 6x + x + 3 = 0$

$(2x+1)(x+3) = 0 \Rightarrow x = -\tfrac{1}{2}, -3$

If $x = -\tfrac{1}{2} \Rightarrow 2y^2 = 3(-\tfrac{1}{2}+1) = \tfrac{3}{2}$

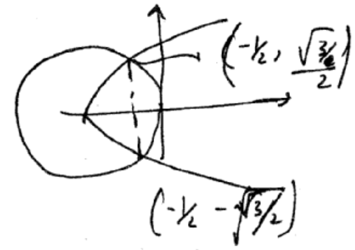
$\Rightarrow \cancel{2y = \pm\sqrt{3/2}} \quad y = \pm \sqrt{3}/2$

If $x = -3 \Rightarrow y^2 = 3(-3+1)$ (not possible)

So intersection points are $(-\tfrac{1}{2}, \pm\sqrt{3/2})$

Length of common chord $= 2\cancel{\sqrt{3/2}}\sqrt{3}/2$

$\Rightarrow \sqrt{3}$



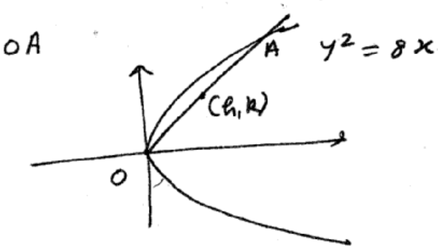
(51) Let $Q(h,k)$ is middle point of OA

$\Rightarrow A = (2h, 2k)$

$(2h, 2k)$ lies on the curve

$\Rightarrow 4k^2 = 8 \cdot 2h$

$\Rightarrow y^2 = 4x$



(54) Let slope of chord be m

Eqn. of focal chord $y = m(x-4) \Rightarrow mx - y - 4m = 0$

is tangent to given circle.

Applying $p = r$

$$\frac{|6m-4m|}{\sqrt{1+m^2}} = \sqrt{2} \Rightarrow 4m^2 = 2(1+m^2)$$

$m = \pm 1$

(55) Let P be $(at^2, 2at)$

Eqn of tangent PT : $ty = x + at^2$

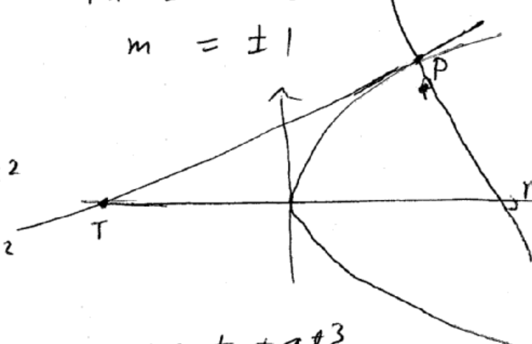
for T, $y = 0 \Rightarrow x = -at^2$

$\Rightarrow$ T is $(-at^2, 0)$

Eqn of normal to P : $y = -tx + 2at + at^3$

for N, $y = 0 \Rightarrow x = 2a + at^2$

$\Rightarrow$ N is $(2a + at^2, 0)$



Let (h,k) is centroid of triangle PTN

$$\Rightarrow h = \frac{at^2 - at^2 + 2a + at^2}{3} = \frac{2a + at^2}{3} \quad (1)$$

(11)

$k = \dfrac{0+0+2at}{3} \Rightarrow k = \dfrac{2at}{3} \Rightarrow t = \dfrac{3k}{2a}$ sub in (1)

$\Rightarrow h = \dfrac{2a}{3} + \dfrac{a}{3}\left(\dfrac{3k}{2a}\right)^2$

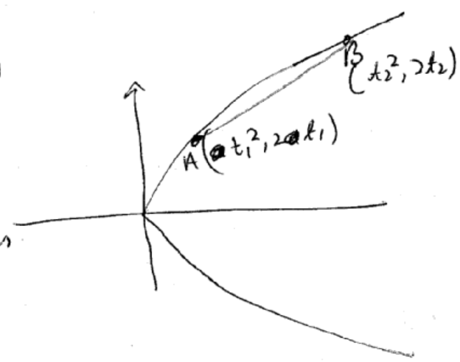
$\Rightarrow h = \dfrac{2a}{3} + \dfrac{3}{4a}k^2$

locus is $x = \dfrac{2a}{3} + \dfrac{3}{4a}y^2$

$\Rightarrow \dfrac{3}{4a}y^2 = x - \dfrac{2a}{3} \Rightarrow y^2 = \dfrac{4a}{3}\left(x - \dfrac{2a}{3}\right)$

Now verify the alternatives.

(56) let A & B be $(t_1^2, 2t_1)$ & $(t_2^2, 2t_2)$



$m_{AB} = \dfrac{2(t_2 - t_1)}{t_2^2 - t_1^2} = \dfrac{2}{t_1+t_2}$

Circle ~~AP~~ with AB as diameter touches the axis ie x axis.

$\Rightarrow |\beta| = r$

$\Rightarrow \left|\left(\dfrac{2t_1+2t_2}{2}\right)\right| = r \Rightarrow |t_1+t_2| = r$

$\Rightarrow t_1+t_2 = \pm r$

$m_{AB} = \dfrac{2}{\pm r} = \pm\dfrac{2}{r}$

(57) (2,0) & (3,0) lies on the curve.

Eqn of tangent at (2,0) $\Rightarrow \frac{1}{2}(y+0) = 2x - \frac{5}{2}(x+2) + 6$

$\Rightarrow \dfrac{y}{2} = -\dfrac{x}{2} + 1$

$\Rightarrow y = -x + 2$ — (1)

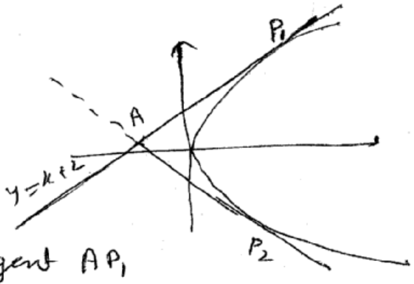
Eqn of tangent at (3,0) $\Rightarrow \frac{1}{2}(y+0) = 3x - \frac{5}{2}(x+3) + 6$

$\dfrac{y}{2} = \dfrac{x}{2} - \dfrac{3}{2}$

$\Rightarrow y = x - 3$ — (2)

Soln. (1) & (2) are $\perp$ $\Rightarrow$ angle is 90°

(58)



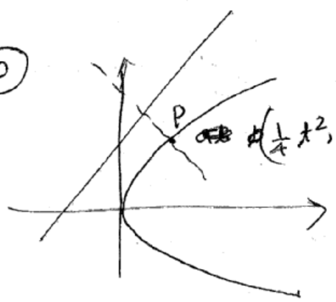
First tangent is AP_1 ($y = x+2$)
Other tangent AP_2 ⊥ to AP_1 then
A point should lie on the directrix.
so A is intersection of directrix & tangent AP_1

Eqn of directrix $x + 2 = 0$
$\Rightarrow x = -2$ sub. in AP_1
$y = -2+2 \Rightarrow y = 0$ so A is $(-2,0)$

(59) Eqn. of normal is $y = -tx + 2t + t^3$ — (1)
passes through $(9,6)$ $\Rightarrow 6 = -9t + 2t + t^3$
$\Rightarrow t^3 - 7t - 6 = 0$
$\Rightarrow t = -1, -2, 3$

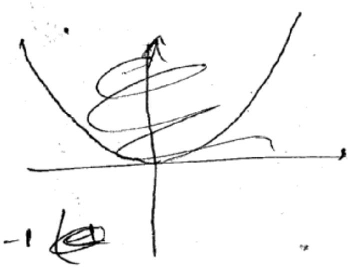
Subsitute the value of t in (1)

(60)



$P\left(\frac{1}{4}t^2, \frac{1}{2}t\right)$

Shortest distance is along normal to both curve.



so slope of line is -1
(⊥ to given line)

Eqn of normal to parabola $y = -tx + 2at + at^3$

$\Rightarrow -t = -1$
$\Rightarrow t = 1$

so P point is $\left(\frac{1}{4}\times 1^2, \frac{1}{2}\times 1\right) = \left(\frac{1}{4}, \frac{1}{2}\right)$

Shortest distance ⊥ distance from $\left(\frac{1}{4}, \frac{1}{2}\right)$ to line

$\Rightarrow d = \dfrac{\left|\frac{1}{2} - \frac{1}{4} - 1\right|}{\sqrt{1+1}} = \dfrac{3}{4\sqrt{2}}$