DISHA CLASSES
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Inverse trigonometric functions

1

Part A

(2) $\cot^{-1}\cot\left(\frac{5\pi}{4}\right) = \cot^{-1}\cot(\pi + \pi/4)$

$= \cot^{-1}\cot(\pi/4) = \pi/4$

(4) (i) $\cos(\tan^{-1} 3/4) = \cos\left(\cos^{-1}\frac{4}{5}\right) = 4/5$

(5) $\cos(2\tan^{-1}\frac{1}{7}) = \cos\left(\tan^{-1}\frac{\frac{2}{7}}{1-(1/7)^2}\right) = \cos\left(\tan^{-1}\frac{7}{24}\right) = \cos\cos^{-1}\left(\frac{24}{25}\right)$

$= \frac{24}{25}$

$\sin(4\tan^{-1}1/3) = \sin\left[2\times 2\tan^{-1}1/3\right] = \sin\left(2\tan^{-1}\frac{2/3}{1-(1/3)^2}\right)$

$= \sin\left(2\tan^{-1}\frac{3}{4}\right) = \sin\left(\tan^{-1}\frac{3/4\times 2}{1-(3/4)^2}\right)$

$= \sin\left(\tan^{-1}\frac{24}{7}\right) = \sin\sin^{-1}\left(\frac{24}{25}\right)$

$= \frac{24}{25}$

Hence L.H.S. = R.H.S.

(6) let $\cos^{-1}\frac{a}{b} = \theta \Rightarrow \cos\theta = \frac{a}{b}$

so L.H.S. is $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \tan\left(\pi/4 - \theta/2\right)$

$= \frac{1+\tan\theta/2}{1-\tan\theta/2} + \frac{1-\tan\theta/2}{1+\tan\theta/2} = \frac{(1+\tan\theta/2)^2 + (1-\tan\theta/2)^2}{(1-\tan\theta/2)(1+\tan\theta/2)}$

$= \frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2} = \frac{2}{\cos\theta} = \frac{2a}{b}$

(7) (i) $2\tan^{-1}1/3 + \tan^{-1}\frac{1}{7}$

$\Rightarrow \tan^{-1}\frac{2/3}{1-(1/3)^2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{6}{8} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \tan^{-1}1 = \pi/4$

(8) $\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \tan\alpha$

Applying componendo & dividendo rule.

$\Rightarrow \frac{2\sqrt{1+x^2}}{-2\sqrt{1-x^2}} = \frac{\tan\alpha+1}{\tan\alpha-1}$

$\Rightarrow \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} = \frac{1+\tan\alpha}{1-\tan\alpha} \Rightarrow \frac{1+x^2}{1-x^2} = \frac{(1+\tan\alpha)^2}{(1-\tan\alpha)^2}$

Again applying componendo & dividendo rule.

$$\frac{2}{2x^2} = \frac{(1+\tan\alpha)^2 + (1-\tan\alpha)^2}{(1+\tan\alpha)^2 - (1-\tan\alpha)^2}$$

$$\Rightarrow \frac{1}{x^2} = \frac{2(1+\tan^2\alpha)}{4\tan\alpha} = \frac{1+\tan^2\alpha}{2\tan\alpha} =$$

$$\Rightarrow x^2 = \frac{2\tan\alpha}{1+\tan^2\alpha} = \sin 2\alpha$$

9(ii) $\tan^{-1}\frac{1}{x+y} + \tan^{-1}\frac{y}{1+x(x+y)}$

$\tan^{-1}\frac{1}{x+y} + \tan^{-1}\frac{y+x-x}{1+x(y+x)}$

$\tan^{-1}\frac{1}{x+y} + \tan^{-1}(y+x) - \tan^{-1}x$

$\cot^{-1}(x+y) + \tan^{-1}(y+x) - \tan^{-1}x = \pi/2 - \tan^{-1}x = \cot^{-1}x$

Q11 $T_r = \tan^{-1}\frac{x}{1+r(r+1)x^2}$

$$= \tan^{-1}\frac{(r+1)x - rx}{1+rx(r+1)x} = \tan^{-1}(r+1)x - \tan^{-1}rx$$

Given quantity is $T_1 + T_2 + T_3 + \text{---} \text{---} T_n$

$= (\tan^{-1}2x - \tan^{-1}x) + (\tan^{-1}3x - \tan^{-1}2x) + (\tan^{-1}4x - \tan^{-1}3x)$

$+ \text{---} \text{---} (\tan^{-1}(n+1)x - \tan^{-1}nx)$

$= -\tan^{-1}x + \tan^{-1}(n+1)x = \tan^{-1}\frac{(n+1)x - x}{1+(n+1)x\cdot x}$

12(ii) $\sin^{-1}(1-x) - 2\sin^{-1}x = \pi/2$

$\Rightarrow -2\sin^{-1}x = \pi/2 - \sin^{-1}(1-x) = \cos^{-1}(1-x)$

$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x)$

$\Rightarrow \cos(-2\sin^{-1}x) = \cancel{\cos^{-1}(}1-x$

$\Rightarrow \cos(2\sin^{-1}x) = 1-x$

$\Rightarrow \cancel{2\sin^2\sin}\ 1-2\sin^2(\sin^{-1}x) = 1-x$

$\Rightarrow 1-2x^2 = 1-x \Rightarrow x = 0 \text{ or } \frac{1}{2}$

but $x=0$, only satisfies the original eqn. so it has only one solution $x=0$

2

14(i) $\tan\left(2\tan^{-1}\frac{1}{\sqrt{5}}\,a - \frac{\pi}{4}\right)$

let $\tan^{-1}\frac{1}{\sqrt{5}} = \theta \Rightarrow \tan\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow \tan\left(2\theta - \pi/4\right) = \frac{\tan 2\theta - 1}{1 + (\tan 2\theta)\cdot 1}$ — (1)

if $\tan\theta = \frac{1}{\sqrt{5}}$ & $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\frac{1}{\sqrt{5}}}{1-\frac{1}{5}}$

$= \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2}$

subsituting in (1) ~~⇒ given~~

$\Rightarrow$ given quantity $= \frac{\frac{\sqrt{5}}{2} - 1}{1 + \frac{\sqrt{5}}{2}} = \frac{\sqrt{5}-2}{\sqrt{5}+2}$

(16) $\sin\left[\cot^{-1}\left\{\cos\left(\tan^{-1}x\right)\right\}\right] = \sin\left[\cot^{-1}\left\{\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}\right]$

$= \sin\left[\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \sin\left[\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2+1}}\right] = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

(15) $\cos\left(2\cos^{-1}x + \sin^{-1}x\right) = \cancel{\cos\left(\pi/2 + \cos\right)}$

$\cos\left(\overline{\cos^{-1}x + \sin^{-1}x} + \cos^{-1}x\right) = \cos\left(\pi/2 + \cos^{-1}x\right)$

$= -\sin\cos^{-1}x = -\sin\sin^{-1}\sqrt{1-x^2}$

$= -\sqrt{1-x^2} = -\sqrt{1-\left(\frac{1}{5}\right)^2}$

(18) if $0 < A < \pi/4 \Rightarrow \cot A > 1$

so $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \pi + \tan^{-1}\frac{\cot A + \cot^3 A}{1 - \cot A\cot^3 A}$, $(\cot^4 A > 1)$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \pi + \tan^{-1}\frac{\cot A\cancel{(1+\cot^2 A)}}{(1-\cot^2 A)\cancel{(1+\cot^2 A)}}$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \pi + \tan^{-1}\left(\frac{\frac{1}{\tan A}}{1 - \frac{1}{\tan^2 A}}\right)$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \pi + \tan^{-1}\left(-\frac{\tan A}{1-\tan^2 A}\right)$

$= \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \pi - \tan^{-1}\left(\frac{1}{2}\tan 2A\right) = \pi$

If $\pi/4 < A < \pi/2 \Rightarrow 0 < \cot A < 1$

$\Rightarrow (\cot A)(\cot^3 A) < 1$

so

$$\tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \tan^{-1}\left(-\frac{\tan A}{1-\tan^2 A}\right)$$

$$= -\tan^{-1}\left(\tfrac{1}{2}\tan 2A\right)$$

& hence required quantity is zero.

(19) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \pi/4$

$\tan^{-1}\frac{x-1}{x-2} = \tan^{-1}1 - \tan^{-1}\frac{x+1}{x+2}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\dfrac{1-\left(\frac{x+1}{x+2}\right)}{1+\left(\frac{x+1}{x+2}\right)}$

$\Rightarrow \frac{x-1}{x-2} = \frac{x+2-(x+1)}{x+2+x+1}$

$\Rightarrow \frac{x-1}{x-2} = \frac{1}{2x+3} \Rightarrow (x-1)(2x+3) = x-2$

$\Rightarrow 2x^2+x-3 = x-2$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

(20) $\sin\left[2\cos^{-1}\{\cot(2\tan^{-1}x)\}\right] = 0$

$\sin\left[2\cos^{-1}\left\{\cot\left(\tan^{-1}\frac{2x}{1-x^2}\right)\right\}\right] = 0 = \sin\left[2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$

$\Rightarrow 2\sin\left[\cos^{-1}\frac{1-x^2}{2x}\right]\cos\left[\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$

$\Rightarrow 2\sin\left[\sin^{-1}\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right]\left[\frac{1-x^2}{2x}\right] = 0$

$\Rightarrow \sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\left[\frac{1-x^2}{2x}\right] = 0 \Rightarrow 1-x^2 = 0$ or $1-\left(\frac{1-x^2}{2x}\right)^2 = 0$

$\Rightarrow x = \pm 1$ or $\left(\frac{1-x^2}{2x}\right) = \pm 1$

$\Rightarrow$ " $1-x^2 = \pm 2x$

$\Rightarrow$ " or $1-x^2 = 2x$ & $1-x^2 = -2x$

Find all the values of x.

(8)

Q21 $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

$\tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3$

$\tan^{-1}\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = \tan^{-1}3 \Rightarrow \frac{xy+1}{y-x} = 3$

$\Rightarrow xy+1 = 3y-3x$

$\Rightarrow \cancel{xy+1}\; y(x-3) = -3x-1$

$\Rightarrow y = \frac{-3x-1}{x-3} = \frac{3x+1}{3-x}$

Since x & y are +ive integer. $\Rightarrow x \in \{0,1,2\}$

if $x=0 \Rightarrow y = \frac{1}{3}$ (not integer)

$x = 1 \Rightarrow y = 2$ ✓

$x = 2 \Rightarrow y = 7$ ✓

so two pairs of (x,y) $(1,2)$ & $(2,7)$

Q22 $y = \cot^{-1}\sqrt{\cos 2x} - \tan^{-1}\sqrt{\cos 2x}$

$= \tan^{-1}\frac{1}{\sqrt{\cos 2x}} - \tan^{-1}\sqrt{\cos 2x}$

$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos 2x}} - \sqrt{\cos 2x}}{1+\frac{1}{\sqrt{\cos 2x}}\sqrt{\cos 2x}}\right) = \tan^{-1}\left(\frac{1-\cos 2x}{2\sqrt{\cos 2x}}\right)$

$\Rightarrow \tan y = \frac{1-\cos 2x}{2\sqrt{\cos 2x}}$

$\Rightarrow \sin y = \frac{1-\cos 2x}{\sqrt{(1-\cos 2x)^2 + 4\cos 2x}} = \frac{1-\cos 2x}{\sqrt{(1+\cos 2x)^2}}$

$= \frac{1-\cos 2x}{1+\cos 2x}$

$= \tan^2 x$

~~Alternative~~. Part B (multiple choice)

(1) let $\cos^{-1}\frac{1}{8} = \theta \Rightarrow \cos\theta = \frac{1}{8}$

$\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\theta/2$

Since $\cos\theta = \frac{1}{8}$

$2\cos^2\theta/2 - 1 = \frac{1}{8} \Rightarrow \cos\theta/2 = \pm\frac{3}{4}$

Since $0 < \theta < \pi/2 \Rightarrow 0 < \theta/2 < \pi/4$ so $\cos\theta/2$ is +ive

$\Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \frac{3}{4}$

(2) $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$\tan^{-1}\frac{x-1+x+1}{1-(x-1)(x+1)} = \tan^{-1}\frac{3x-x}{1+3x\cdot x}$

$\Rightarrow \frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2} \Rightarrow x = 0$ or

$\frac{1}{2-x^2} = \frac{1}{1+3x^2}$

$\Rightarrow 2-x^2 = 1+3x^2$

$x = \pm\frac{1}{2}$

(3) (a) ~~$\tan^{-1}|\tan x$~~ $\tan|\tan^{-1}x| = \begin{cases}\tan(-\tan^{-1}x), & x \le 0\\ \tan(\tan^{-1}x), & x \geq 0\end{cases}$

$= \begin{cases}-x, & x \le 0\\ x, & x \geq 0\end{cases} = |x|$ so (a) is correct

(b) $\cot|\cot^{-1}x| = \cot(\cot^{-1}x)$ (because $0 \le \cot^{-1}x < \pi$

$= x$ so (b) is incorrect

(c) Since $\tan^{-1}\tan x \neq x$ (for any value of x) so (c) is incorrect.

(d) is correct

(4) $\cos^{-1}\left\{\cos\left(-\frac{17\pi}{15}\right)\right\} = \cos^{-1}\cos\left(\frac{17\pi}{15}\right) = \cos^{-1}\cos\left(2\pi - \frac{13\pi}{15}\right)$

$= \cos^{-1}\cos\frac{13\pi}{15} = \frac{13\pi}{15}$

(6) Since $d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$

so $\tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{a_4-a_3}{1+a_4a_3}\right) + \cdots \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)\right]$

$\tan\left[(\tan^{-1}a_2 - \tan^{-1}a_1) + (\tan^{-1}a_3 - \tan^{-1}a_2) + (\tan^{-1}a_4 - \tan^{-1}a_3) + \cdots + (\tan^{-1}a_n - \tan^{-1}a_{n-1})\right]$

$= \tan\left[-\tan^{-1}a_1 + \tan^{-1}a_n\right] = \tan\left[\tan^{-1}\frac{a_n-a_1}{1+a_na_1}\right] = \frac{a_n-a_1}{1+a_na_1} = \frac{(n-1)d}{1+a_1a_n}$

(4)

(6) $\tan\left(2\tan^{-1}\frac{1}{5} - \pi/4\right) = \tan\left[\tan^{-1}\frac{2/5}{1-(1/5)^2} - \pi/4\right]$

$= \tan\left[\tan^{-1}\frac{5}{12} - \pi/4\right]$

$= \dfrac{\tan(\tan^{-1} 5/12) - \tan \pi/4}{1 + \tan(\tan^{-1} 5/12)\cdot \tan \pi/4}$

$= \dfrac{\frac{5}{12} - 1}{1 + \frac{5}{12}} = -\frac{7}{17}$

(7) $\sin^{-1}x + \sin^{-1}y = 2\pi/3$

$\Rightarrow \pi/2 - \cos^{-1}x + \pi/2 - \cos^{-1}y = 2\pi/3 \Rightarrow \cos^{-1}x + \cos^{-1}y = \pi/3$

(8) $f(x) = e^{\cos^{-1}\sin(x+\pi/3)}$

let $h(x) = \cos^{-1}\sin(x+\pi/3)$

$h\left(\frac{8\pi}{9}\right) = \cos^{-1}\sin\left(\frac{8\pi}{9} + \pi/3\right) = \cos^{-1}\sin\left(\frac{11\pi}{9}\right)$

$= \pi/2 - \sin^{-1}\sin\frac{11\pi}{9}$

$= \pi/2 - \sin^{-1}\sin(\pi + 2\pi/9)$

$= \pi/2 - \sin^{-1}[-\sin 2\pi/9]$

$= \pi/2 - \sin^{-1}\sin(-2\pi/9)$

$= \pi/2 - (-2\pi/9) = \frac{13\pi}{18}$

Similarly $h\left(\frac{-7\pi}{4}\right) = \cos^{-1}\sin\left(-\frac{7\pi}{4} + \pi/3\right) = \cos^{-1}\sin\left(\frac{-17\pi}{12}\right)$

$= \pi/2 - \sin^{-1}\sin\left(\frac{-17\pi}{12}\right) = \pi/2 + \sin^{-1}\sin\frac{17\pi}{12}$

$= \pi/2 + \sin^{-1}\sin\left(\pi + \frac{5\pi}{12}\right)$

$= \pi/2 + \sin^{-1}\left[-\sin\frac{5\pi}{12}\right]$

$= \pi/2 - \sin^{-1}\sin 5\pi/12 = \pi/2 - \frac{5\pi}{12}$

$= \pi/12$

(9) $\cos^{-1}\left[x\cdot\frac{y}{2} - \sqrt{1-x^2}\sqrt{1-y^2/4}\right] = \alpha$

$\Rightarrow \frac{xy}{2} - \sqrt{1-x^2}\,\frac{\sqrt{4-y^2}}{2} = \cos\alpha$

$\Rightarrow (xy - 2\cos\alpha) = \sqrt{1-x^2}\sqrt{4-y^2}$

$\Rightarrow (xy - 2\cos\alpha)^2 = (1-x^2)(4-y^2)$

$\Rightarrow \cancel{x^2y^2} + 4\cos^2\alpha - 4xy\cos\alpha = 4 - 4x^2 - y^2 + \cancel{x^2y^2}$

$\Rightarrow 4x^2 + 4y^2 - 4xy\cos\alpha = 4 - 4\cos^2\alpha = 4\sin^2\alpha$

(11) $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$

$\Rightarrow \sin^{-1}x_1 + \sin^{-1}x_2 + \sin^{-1}x_3 + \sin^{-1}x_3 + \text{---} + \sin^{-1}x_{2n} = n\pi$

Since $\sin^{-1}x_i \le \pi/2$

$\Rightarrow \sum_{i=1}^{2n} \sin^{-1}x_i \le \sum_{i=1}^{2n} \pi/2 = n\pi$

$\Rightarrow \sum_{i=1}^{2n} \sin^{-1}x_i \le n\pi$

but it is given $\sum_{i=1}^{2n} \sin^{-1}x_i = n\pi$

so $\sin^{-1}x_1 = \sin^{-1}x_2 = \sin^{-1}x_3 = \text{---} = \sin^{-1}x_{2n} = \pi/2$

$\Rightarrow x_1 = x_2 = x_3 = \text{----} = x_{2n} = 1$

so $\sum_{i=1}^{2n} x_i = 2n$

(12) $\tan\left(\tan^{-1}\frac{7}{1} - \tan^{-1}\frac{4}{1}\right) = \tan\left(\tan^{-1}\frac{7-4}{1+7\cdot 4}\right) = \tan\tan^{-1}\left(\frac{3}{29}\right)$
$= 3/29$

(13) $\sqrt{1-\sin x} = |\sin x/2 - \cos x/2|$ & $\sqrt{1+\sin x} = |\sin x/2 + \cos x/2|$

$\cot^{-1}\left\{\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} - \sqrt{1+\sin x}}\right\} = \cot^{-1}\left[\frac{|\sin x/2 - \cos x/2| + |\sin x/2 + \cos x/2|}{|\sin x/2 - \cos x/2| - |\sin x/2 + \cos x/2|}\right]$

if $\pi/2 < x < \pi \Rightarrow \pi/4 < x/2 < \pi/2$ so $\sin x/2 > \cos x/2 > 0$

$\Rightarrow \cot^{-1}\left[\frac{\sin x/2 - \cancel{\cos x/2} + \sin x/2 + \cancel{\cos x/2}}{\cancel{\sin x/2} - \cos x/2 - (\cancel{\sin x/2} + \cos x/2)}\right] = \cot^{-1}\left[\frac{2\sin x/2}{-2\cos x/2}\right]$

$= \cot^{-1}(-\tan x/2)$

$= \cot^{-1}\cot(\pi/2 + x/2) = \pi/2 + x/2$

(14) Similar to Q21 subjective.

(15) $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} = \tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)$

$= \tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{x/y - 1}{1+1\cdot x/y}\right)$

$= \tan^{-1}\frac{x}{y} - \left[\tan^{-1}x/y - \tan^{-1}1\right] = \pi/4$

(18) $3\sin x - 4\sin^3 x = \sin 3x$

The largest interval in which $\sin\theta$ is increasing is $[-\pi/2, \pi/2]$

so length $= \pi$

$\Rightarrow$ length of $3x = \pi$

$\Rightarrow$ length of $x = \pi/3$

(19) $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^4}{4} \dots \infty\right) = \pi/2 - \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} \dots \infty\right)$

$= \sin^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - - \infty\right)$

$\Rightarrow x - \frac{x^2}{2} + \frac{x^4}{4} + - \infty = x^2 - \frac{x^4}{2} + \frac{x^6}{4} + - \infty$

$\Rightarrow \frac{x}{1-(-x/2)} = \frac{x^2}{1-(-x^2/2)}$ (Sum of infinite G.P.)

$\Rightarrow \frac{x}{1+x/2} = \frac{x^2}{1+x^2/2} \Rightarrow x = 0$ or $\frac{1}{1+x/2} = \frac{x}{1+x^2/2}$

$\Rightarrow 1 + x^2/2 = x + x^2/2$

$\Rightarrow x = 1$

(20) $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \pi/2$

For $\tan^{-1}\sqrt{x(x+1)}$ to be defined $x(x+1) \geq 0 \Rightarrow x^2+x \geq 0$ — (1)

For $\sin^{-1}\sqrt{x^2+x+1}$ to be defined $0 \leq x^2+x+1 \leq 1 \Rightarrow -1 \leq x^2+x \leq 0$ (2)

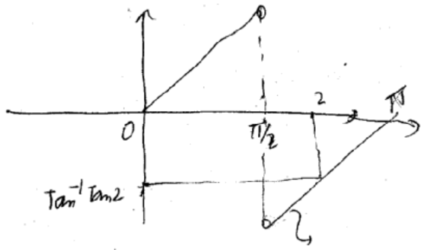
if (1) & (2) are satisfied $\Rightarrow x^2+x = 0$

$\Rightarrow x = 0, -1$

So left hand side terms in the equation exist only when $x = 0, -1$

both of these satisfies to equation so they are solution of equation

(21) Graph of $y = \tan^{-1}\tan x$



From graph $\tan^{-1}\tan 2 = 2-\pi$

$\cos(2-\pi) = \cos(\pi-2) = -\cos 2$

Eqn of line $y-0 = 1(x-\pi)$

when $x = 2$

$\Rightarrow y = 2-\pi$

(22) $\sec^{-1}x = \operatorname{cosec}^{-1}y$

$\Rightarrow \cos^{-1}\frac{1}{x} = \sin^{-1}\frac{1}{y} = \frac{\pi}{2} - \cos^{-1}\frac{1}{y}$

$\Rightarrow \cos^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{y} = \frac{\pi}{2}$

(23) $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$

$\Rightarrow \cos^{-1}\left[\sqrt{p}\sqrt{1-p} - \sqrt{1-(\sqrt{p})^2}\sqrt{1-(\sqrt{1-p})^2}\right] + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$

$\Rightarrow \cos^{-1}0 + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$

$\Rightarrow \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$

$\Rightarrow \sqrt{1-q} = \frac{1}{\sqrt{2}}$

(24) $\sin^{-1}x = 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$

$\Rightarrow x = \frac{2x}{1+x^2} \Rightarrow x = 0$ or $1 = \frac{2}{1+x^2} \Rightarrow x^2 = 1$, $x = \pm 1$

(25) Example - 19

(26) $\cos^{-1}x - \sin^{-1}x = \sin^{-1}(1-x)$

$\frac{\pi}{2} - \sin^{-1}x - \sin^{-1}x = \sin^{-1}(1-x) \Rightarrow \frac{\pi}{2} - 2\sin^{-1}x = \sin^{-1}(1-x)$

$\Rightarrow \sin\left(\frac{\pi}{2} - 2\sin^{-1}x\right) = 1-x$

$\Rightarrow \cos(2\sin^{-1}x) = 1-x$

$\Rightarrow 1 - 2\sin^2(\sin^{-1}x) = 1-x$

$\Rightarrow 1 - 2x^2 = 1-x$

$\Rightarrow x = 0, \frac{1}{2}$

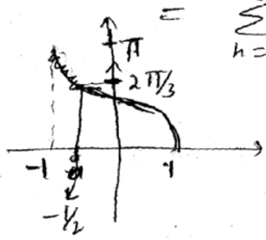
both satisfies the Eqn. so both are solution.

(27) Subjective Q(6)

(28) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{1+n^2+n} = \sum_{n=1}^{\infty} \tan^{-1}\frac{n+1-n}{1+n(n+1)}$

$\displaystyle = \sum_{n=1}^{\infty}\left[\tan^{-1}(n+1) - \tan^{-1} 1\right] = \tan^{-1}\infty - \tan^{-1} 1$

$= \pi/2 - \pi/4 = \pi/2$

(29) 

$\rightarrow$ Graph of $y = \cos^{-1}x$

from graph $\cos^{-1}x > 2\pi/3$

$\Rightarrow -1 \le x < -\frac{1}{2}$

so $-1 \le \frac{n}{2\pi} < -\frac{1}{2}$

$\Rightarrow -2\pi \le n \le -\pi \quad \Rightarrow \quad -6.28 \le n \le -3.14$

Hence least & greatest integral values of n are -6 & -4

Q30 $\rightarrow$ Solved example 18

Q31 $\cos\left(2\tan^{-1}\frac{3}{4}\right)$

Let $\tan^{-1}3/4 = \theta \Rightarrow \tan\theta = 3/4$

$\cos\left(2\tan^{-1}3/4\right) = \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\frac{9}{16}}{1+\frac{9}{16}} = \dfrac{7}{25}$

Similarly $\sin\left(2\cot^{-1}\frac{1}{2}\right) = \sin\left(2\tan^{-1}2\right)$

$= \dfrac{2\tan(\tan^{-1}2)}{1+\tan^2(\tan^{-1}2)} = \dfrac{2\times 2}{1+2^2} = \dfrac{4}{5}$

So given quantity is $\tan^{-1}\left(\frac{7}{25}+\frac{4}{5}\right) = \tan^{-1}\frac{27}{25} > \tan^{-1}1 = \pi/4$

(32) $3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}1/x = \tan^{-1}\frac{1}{3}$

$3\tan^{-1}(2-\sqrt{3}) - \tan^{-1}(1/x) = \tan^{-1}1/3$

$\Rightarrow 3\times\frac{\pi}{12} - \tan^{-1}1/x = \tan^{-1}\frac{1}{3}$

$\Rightarrow \frac{\pi}{4} - \tan^{-1}1/3 = \tan^{-1}1/x$

$\Rightarrow \tan^{-1}1 - \tan^{-1}(1/3) = \tan^{-1}1/x$

$\Rightarrow \tan^{-1}\dfrac{1-1/3}{1+1/3} = \tan^{-1}x \Rightarrow x = \frac{1}{2}$

<u>Q33</u> <u>Subjective Q8</u>

(34) $\alpha+\beta = \underbrace{\sin^{-1}\sqrt{3}/2 + \overbrace{\sin^{-1}1/3 + \cos^{-1}\sqrt{3}/2}} + \cos^{-1}1/3 = \pi/2 + \pi/2 = \pi$

$\alpha = \sin^{-1}\sqrt{3}/2 + \sin^{-1}1/3 = \pi/3 + \sin^{-1}1/3$

also $\sin^{-1}(1/2) > \sin^{-1}(1/3)$ ($\sin^{-1}x$ is increasing function of x)

$\Rightarrow \pi/6 > \sin^{-1}1/3$

Since $\sin^{-1}1/3 < \pi/6 \Rightarrow \sin^{-1}\sqrt{3}/2 + \sin^{-1}1/3 < \pi/3 + \pi/6$
$< \pi/2$

so $\alpha < \pi/2$, $\alpha+\beta = \pi \Rightarrow \beta > \pi/2$

hence $\alpha < \beta$

(36) $\cot^{-1}\left(\frac{n}{\pi}\right) > \pi/6 = \cot^{-1}(\sqrt{3})$

$\Rightarrow \cot^{-1}\left(\frac{n}{\pi}\right) > \cot^{-1}(\sqrt{3})$

$\Rightarrow \frac{n}{\pi} < \sqrt{3}$ ($\cot^{-1}x$ is decreasing function of x)

$\Rightarrow n < \sqrt{3}\pi$

$\sqrt{3}\pi$ is less than 6 so maximum natural no. n is 5.

(37) $\cos^{-1}x \geq 0$ so $\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right| < \pi/3$

$\Rightarrow \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) < \pi/3$

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) < \cos^{-1}1/2$

$\Rightarrow \frac{1-x^2}{1+x^2} > \frac{1}{2}$ ($\cos^{-1}x$ is decreasing function of x)

$\Rightarrow 2-2x^2 > 1+x^2$

$\Rightarrow 3x^2 < 1 \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

(40) $\sin(2\sin^{-1}0.8) = 2\sin\sin^{-1}(0.8)\cos(\sin^{-1}0.8)$

$= 2(0.8)\cos\cos^{-1}\sqrt{1-(0.8)^2}$

$= 2(0.8)(0.6)$

(41) $B = 3\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{3}{5}$

$= \sin^{-1}\left[3\times\frac{1}{3} - 4\times\left(\frac{1}{3}\right)^3\right] + \sin^{-1}\frac{3}{5}$

$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\frac{3}{5}$

$= \tan^{-1}\frac{23}{10\sqrt{2}} + \tan^{-1}\left(\frac{3}{4}\right)$

Since $2\sqrt{2}-1 > \frac{23}{10\sqrt{2}}$ & $2\sqrt{2}-1 > \frac{3}{4}$

So $\tan^{-1}(2\sqrt{2}-1) > \tan^{-1}\frac{23}{10\sqrt{2}}$ & $\tan^{-1}(2\sqrt{2}-1) > \tan^{-1}(3/4)$

So $2\tan^{-1}(2\sqrt{2}-1) > \tan^{-1}\left(\frac{23}{10\sqrt{2}}\right) + \tan^{-1}\frac{3}{4}$

$\cancel{= \pi + \tan^{-1}\frac{\frac{23}{10}+\frac{3}{4}}{1-\frac{23}{10}\cdot\frac{3}{4}}}$

$\cancel{= \pi + \tan^{-1}\left(-\frac{122}{29}\right)}$

$\cancel{= \pi - \tan^{-1}\left(\frac{122}{29}\right)}$

Q42 → Subjective Q.16

Comprehension

For $\cos^{-1}x$ & $\sin^{-1}(x-2)$ to be defined, $-1\le x\le 1$ & $-1\le x-2\le 1$

$\Rightarrow -1\le x\le 1$ & $1\le x\le 3$

(a) So there is only one value of x which is common in above condition

So $x = 1$

$\sin(\cos^{-1}x + \sin^{-1}(x-2)) = \sin(\cos^{-1}(-1) + \sin^{-1}(-1)) = \sin\pi/2 = 1$

(b) $\cos\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{3}\right) = \cos\left(\sin^{-1}\frac{1}{2}\right)\cos\left(\cos^{-1}\frac{1}{3}\right) - \sin\left(\sin^{-1}\frac{1}{2}\right)\sin\sin^{-1}\frac{1}{3}$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{\sqrt{8}}{3} = \frac{\sqrt{3}-\sqrt{8}}{6}$

Additional Questions

Q1 $\angle C = 90^\circ \Rightarrow a^2+b^2=c^2$

$\tan^{-1}\left(\frac{\frac{a}{b+c}+\frac{b}{c+a}}{1-\left(\frac{a}{b+c}\right)\left(\frac{b}{c+a}\right)}\right) = \tan^{-1}\left(\frac{a(c+a)+b(b+c)}{(b+c)(c+a)-ab}\right)$

$= \tan^{-1}\left(\frac{a^2+b^2+ac+bc}{bc+\cancel{ab}+c^2+ac-\cancel{ab}}\right)$

$= \tan^{-1}\left(\frac{c^2+ac+bc}{bc+c^2+ac}\right) = \tan^{-1}1 = \pi/4$

Q2 If $x \in (\pi/2, \pi)$ then $\cos^{-1}\cos x = x$ & $\sin^{-1}\sin x = \pi - x$

so $\sin^{-1}[\cos\{x + \pi - x\}] = \sin^{-1}(\cos\pi) = \sin^{-1}(-1) = -\pi/2$

Q4 We know ~~$\cos^{-1}\alpha_1 \geq 0$~~ $0 \leq \cos^{-1}\alpha_i \leq \pi$

so each of $\cos^{-1}\alpha_i \geq 0$

so $\sum_{i=1}^{n} \cos^{-1}\alpha_i = 0$ only when $\cos^{-1}\alpha_1 = \cos^{-1}\alpha_2 = \cos^{-1}\alpha_3$

$= - - \cos^{-1}\alpha_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = - - - = \alpha_n = 1$

so $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} 1 = n$

Q5 $\sin^{-1}1 - \sin^{-1}\sqrt{\frac{3}{x}} - \pi/6 = 0$

$\Rightarrow \pi/2 - \sin^{-1}\sqrt{3/x} - \pi/6 = 0 \Rightarrow \sin^{-1}\sqrt{3/x} = \pi/3$

$\Rightarrow \sqrt{3/x} = \frac{\sqrt{3}}{2} \Rightarrow x = 4$

Q6 $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

for $\sin^{-1}x$ & $\sec^{-1}x$ to be defined

$-1 \leq x \leq 1$ & $x \geq 1$ or $x \leq -1$

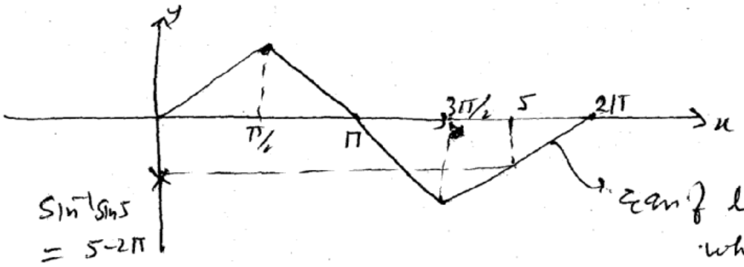
combining these $\Rightarrow x = \pm 1$

so Range consist of only two points

$f(1) = \pi/2 + \pi/4 + 0 = 3\pi/4$

$f(-1) = -\pi/2 - \pi/4 + \pi = \pi/4$

Q7 Graph of $y = \sin^{-1}\sin x$



$\sin^{-1}\sin 5 = 5 - 2\pi$

$\frac{3\pi}{2} = \frac{3}{2} \times \frac{22}{7} = \frac{33}{7} < 5$

Eqn of line $y = 1(x - 2\pi)$

when $x = 5$

$y = 5 - 2\pi$

So $\Rightarrow 5 - 2\pi > x^2 - 4x$

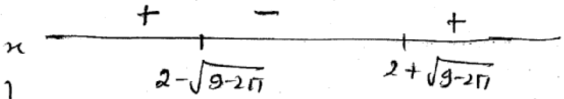
$\Rightarrow x^2 - 4x + 2\pi - 5 < 0$

$\Rightarrow [x - (2 + \sqrt{9 - 2\pi})][x - (2 - \sqrt{9 - 2\pi})] < 0$

$x = \frac{4 \pm \sqrt{16 - 4(2\pi - 5)}}{2}$

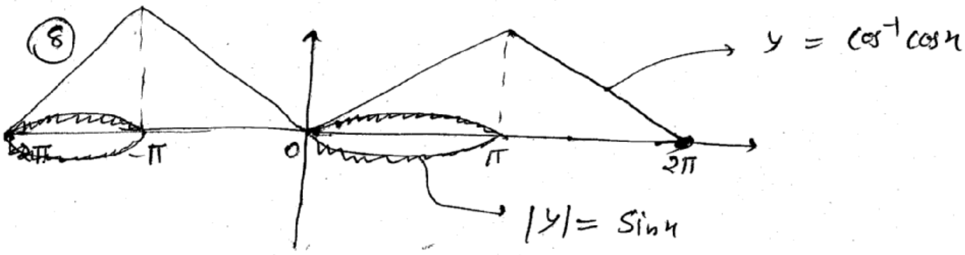
$= 2 \pm \sqrt{9 - 2\pi}$

$\Rightarrow$



$\Rightarrow x \in \left(2-\sqrt{9-2\pi}\ ,\ 2+\sqrt{9-2\pi}\right)$

(8)



From graph both curve intersects at $x = 0, 2\pi, -2\pi$

$\Rightarrow$ 3 point of intersection.

(9) $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \cos\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$= \cos\left(\tan^{-1}\dfrac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) = \frac{17}{6} = \frac{9}{5}$

(a), (b), (c) are correct.

(10) $2x = \tan(2\tan^{-1}a) + 2\tan\left(\tan^{-1}a + \tan^{-1}a^3\right)$

$= \dfrac{2a}{1-a^2} + 2\left(\dfrac{a+a^3}{1-a\cdot a^3}\right), \quad a \neq \pm 1$

$2x = \dfrac{2a}{1-a^2} + \dfrac{2a}{1-a^2}$

$\Rightarrow x = \dfrac{2a}{1-a^2} \Rightarrow x - xa^2 = 2a$

$\Rightarrow xa^2 + 2a = x$

So (a) & (d) are valid $\Rightarrow$ b, c are in valid.

(11) $-1 \le x \le 1$

(a) is correct.

(b) $k^2-2k+3 = (k-1)^2+2 \geq 2$ so $f(x)$ is not defined at $x = k^2-2k+3$

(c) $\frac{1}{1+k^2} < 1$ & $\frac{1}{1+k^2} > 0 \Rightarrow f\left(\frac{1}{1+k^2}\right)$ is valid.

(d) $x = -2$ is not valid $\Rightarrow$ (d) is incorrect.